PRAMANA
— journal of
physics

© Printed in India

Vol. 45, No. 4,
October 1995
pp. 305–309

Painlevé analysis and integrability of the damped anharmonic oscillator equation

S PAUL RAJ and S RAJASEKAR*
Department of Physics, St. Xavier's College, Tirunelveli 627 002, India
*Department of Physics, Manonmaniam Sundaranar University, Tirunelveli 627 002, India

MS received 24 April 1995; revised 9 August 1995

Abstract. The Painlevé analysis is applied to the anharmonic oscillator equation $\ddot{x} + d\dot{x} + Ax + Bx^2 + Cx^3 = 0$. The following three integrable cases are identified: (i) $C = 0$, $d^2 = 25\,A/6$, $A > 0$, B arbitrary, (ii) $d^2 = 9A/2$, $B = 0$, $A > 0$, C arbitrary and (iii) $d^2 = -9A/4$, $C = 2B^2/(9A)$, $A < 0$, $C < 0$, B arbitrary. The first two integrable choices are already reported in the literature. For the third integrable case the general solution is found involving elliptic function with exponential amplitude and argument.

Keywords. Anharmonic oscillator; Painlevé analysis; exact solution.

PACS No. 02·90

1. Introduction

In this paper we study the integrability of the nonlinear system

$$\ddot{x} + d\dot{x} + Ax + Bx^2 + Cx^3 = 0, \tag{1}$$

where an overdot denotes differentiation with respect to time t, d is the damping coefficient, B and C are quadratic and cubic nonlinearities coefficients, and A is the square of the natural frequency of the system. Very recently, chaotic behaviour in (1) subjected to a harmonic parametric excitation is studied [1]. The anharmonic oscillator equation (1) arises in many situations in physics and engineering [2] apart from the application to a classical anharmonic oscillator. The addition of an external periodic forcing term to (1) with $B = 0$ results in the Duffing oscillator and with $C = 0$ gives mechanical oscillator. System (1) with external periodic force and parametric perturbation exhibits complex dynamics [1, 3–5]. Some special analytical solutions of (1) are reported in [2]. The integrability of (1) with $B = 0$ is studied by Euler *et al* [6], Duarte *et al* [7], Parthasarathy and Lakshmanan [8] and Estevez [9].

Our motivation in the present paper is to investigate the integrability of (1) for nonzero values of d, A, B and C. Particularly by applying the Painlevé analysis [10, 11] we show that the system (1) is integrable for the specific choice of the parameter values $d^2 = -9A/4$, $C = 2B^2/(9A)$, $A < 0$ and B arbitrary in addition to the two other integrable choices reported in the literature for special form of (1). For this choice we then give the analytical solution.

2. Painlevé analysis of the damped anharmonic oscillator

The P-property requires that the solutions of (1) may be written as Laurent series expansion in the complex variable [10, 11] $\tau = t - t_0$ with leading order

$$x \sim a_0 \tau^p, \tag{2}$$

where p is yet determined. Inserting the dominant term (2) in (1) one finds two possibilities

(i) $p = -2, \quad a_0 = -6/B, \quad C = 0.$
(ii) $p = -1, \quad a_0^2 = -2/C.$

Now to find higher order terms we write

$$x = a_0 \tau^p + \alpha \tau^{p+r} \tag{3}$$

and substitute in (1) to obtain resonances, that is, conditions such that arbitrary constants may enter in the expansion (3). For the case $p = -2$ resonance condition is $(r+1)(r-6) = 0$. In other words, the resonances occur at $r = -1, 6$. For the case (ii) $(p = -1)$ the resonance values become $r = -1, 4$. The root -1 corresponds to the arbitrariness of t_0.

To verify the occurrence of sufficient number of arbitrary constants we introduce the series expansion

$$x = a_0 \tau^p + \sum_{k=1}^{r} a_k \tau^{p+k} \tag{4}$$

in (1) and equate the coefficients of various powers of τ to zero.

Case (i): Here the resonances are $r = -1, 6$. The values of the coefficients a_i, $i = 1, 2, \ldots, 5$ are given by

$$a_1 = 6d/(5B), \quad a_2 = (d^2 - 25A)/(50B), \quad a_3 = d^3/(250B),$$
$$a_4 = (7d^4 - 125A^2)/(5000B), \quad a_5 = (79d^5 - 1375dA^2)/(75000B).$$

For a_6 we obtain

$$a_6(0) = 25A - 6d^2.$$

Hence, case(i) possesses P-property for

$$C = 0, \quad d^2 = 25A/6, \quad A > 0, \quad B \text{ arbitrary.} \tag{5}$$

Case (ii): Here $p = -1, r = -1, 4$. From the coefficients of $\tau^{-3}, \tau^{-2}, \tau^{-1}, \tau^0$ we obtain

$$a_0^2 = -2/C, \quad a_1 = -(2B + Cda_0)/6C, \quad a_2 = a_0(6AC - 2B^2 - Cd^2)/36C,$$
$$a_3 = da_0(9AC - 3B^2 - 2d^2C)/(108C) + B(2B^2 - 9AC)/(108C^2), \tag{6}$$

respectively. From the coefficients of τ^1 in (1) we get

$$a_4(0) = -2da_3 - 2a_0a_3(B + 3Ca_1) - 3Ca_0a_2^2$$
$$- a_2(A + 2Ba_1 + 3Ca_1^2).$$

Simple manipulation of the above equation yields $a_3 = 0$. From (6) we note that a_3 becomes zero either for $da_0\alpha_1/(108C) + \alpha_2 = 0$ with $\alpha_1 \neq 0, \alpha_2 \neq 0$ or for $\alpha_1 = \alpha_2 = 0$

where

$$\alpha_1 = 9AC - 3B^2 - 2d^2C, \quad \alpha_2 = B(2B^2 - 9AC)/(108C^2). \tag{7}$$

In the next section we show that analytical solution exists for $\alpha_1 = \alpha_2 = 0$, that is, for

$$2B^2 - 9AC = 0, \quad 9AC - 3B^2 - 2d^2C = 0. \tag{8}$$

From (8) the integrable conditions are

$$d^2 = -9A/4, \quad C = 2B^2/(9A), \quad A < 0, \quad B \text{ arbitrary}. \tag{9}$$

Further from (7), the conditions $\alpha_1 = 0, \alpha_2 = 0$ with $B = 0$ gives

$$d^2 = 9A/2, \quad B = 0, \quad A > 0, \quad C \text{ arbitrary}. \tag{10}$$

Thus we find that (1) possesses P-property for three parametric restrictions given by (5), (9) and (10). The third integrable choice (10) is reported in [6, 8, 9]. To the best of our knowledge the integrable choice (9) has not been listed in the literature.

3. Analytical solution for the integrable choice (9)

Analytical solution for the integrable choices (5) and (10) are given in [12, 2, 8]. In this section we obtain the exact solution of (1) for the parametric restrictions (9). For nonlinear differential equations, at present, no general methods are available to find analytical solutions. However, exact analytical solution is reported in the literature for certain form of second-order ordinary differential equation. For example in ref. [12] analytical solution is given for the equation

$$\ddot{W} + AW + BW^3 = 0, \tag{11}$$

where A and B are arbitrary. Since the above equation is a special case of (1), we check the possibility of reducing (1) into (11) by a suitable transformation. This may be possible for specific parametric choices.

First we write

$$x = uV \tag{12}$$

and find V so that no first derivative in u appears. We obtain $V = V_0 e^{-d/2t}$. For convenience we choose $V_0 = 1$. Equation (1) now becomes

$$\ddot{u} + (A - d^2/4)u + Be^{d/2t}u^2 + Ce^{-dt}u^3 = 0. \tag{13}$$

Then we use the transformation linear in W of the form

$$u = \alpha e^{at} + \beta W(Z)e^{gt}, \tag{14a}$$

$$Z = be^{ft}. \tag{14b}$$

When (14a) is substituted in (13) exponential time dependence in the second term of (14a) gives a first derivative in W. To eliminate the first derivative of W the transformation (14b) is introduced. Now we get a differential equation in W. To determine the constants of (14), we substitute (14) in (13) and equate the coefficients of W^0, W^2 and dW/dZ to zero. In the reduced differential equation all the terms contain exponential time dependence except the coefficient of W. So we equate the coefficient of W to zero.

Further when $f = -2g$ the exponential time dependent terms disappear and we obtain

$$d^2W/dZ^2 = W^3. \tag{15}$$

The constants in (14) are found to be

$$a = d/2, \quad f = -d/3, \quad g = d/6, \quad \alpha = -B/3C, \quad \beta = 2d^2/3B, \quad b = \sqrt{2} \tag{16a}$$

and further we require

$$d^2 = -9A/4, \quad C = 2B^2/9A. \tag{16b}$$

We note that the conditions given by (16b) are the parametric choices obtained by P-analysis (clarify eq. (9)). More precisely equation (1) under the transformation

$$x(t) = (2d^2/3B)(1 + W(Z)e^{-(d/3)t}), \tag{17a}$$

$$Z = \sqrt{2}e^{-(d/3)t} \tag{17b}$$

together with the conditions (16b) reduces to (15) which is (11) when $A = 0$.

The first integral of (15) is

$$(dW/dZ)^2 = W^4/2 + c_0, \tag{18}$$

where c_0 is an integration constant. We consider two cases in which $c_0 = 0$ and $c_0 \neq 0$ separately. For $c_0 \neq 0$ with $W(0) = W_0 = (-2c_0)^{1/2}$ and $dW/dZ|_{Z=0} = 0$ real solutions of (15) exist only for $c_0 < 0$ and are given by

$$W = W_0/cn(W_0 v; k), \quad k^2 = 1/2, \quad v = Z - Z_0, \tag{19}$$

where Z_0 is a second integration constant so that the solution of (1) becomes

$$x(t) = (2d^2/3B)[1 + W_0 \exp(-dt/3)/cn(W_0 Z; k)],$$

$$Z = \sqrt{2}\exp(-dt/3) - Z_0. \tag{20}$$

The solution (20) for $d > 0$ exponentially decays to the stable fixed point (node) in the limit $t \to \infty$ for the range $0 < |W_0 Z_0| < \{\pi/2F(1/2; 1/2; 1; 1/2) - 2\}$. (The other two fixed points are saddle.) However, for $c_0 = 0$ the solution given by (20) becomes simple. In this case, straightforward integration of (18) gives the solution of (1) as

$$x(t) = (2d^2/3B)[1 \pm 1/(1 + c_1 \exp(dt/3))],$$

where c_1 is the integration constant.

Acknowledgement

The authors would like to thank the referee for several suggestions. This work forms part of UGC minor research project.

References

[1] J M Malasoma, C H Lamarque and L Jezequel, *Int. J. Nonlinear Dynamics* **5**, 153 (1994)
[2] J M Dixon, J A Tuszynski and M Otwinowski, *Phys. Rev.* **A44**, 3481 (1991)
[3] R H Plaut and J C Hsiech, *J. Sound Vib.* **120**, 73 (1987)

[4] L D Zavodney and A H Nayfeh, *J. Sound Vib.* **120**, 86 (1988)
[5] W Szemplinska-Stupnicka, R H Plaut and J C Hsieh, *J. Appl. Mech.* **56**, 947 (1989)
[6] N Euler, W H Steeb and K Cyrus, *J. Phys.* **A22**, L195 (1989)
[7] L G S Duarte, N Euler and I C Moreira, *J. Phys.* **A23**, 1457 (1990)
[8] S Parthasarathy and M Lakshmanan, *J. Sound Vib.* **137**, 523 (1990)
[9] P G Estevez, *J. Phys.* **A24**, 1153 (1991)
[10] M J Ablowitz, A Ramani and H Segur, *J. Math. Phys.* **21**, 715 (1980)
[11] A Ramani, B Grammaticos and T Bountis, *Phys. Rep.* **180**, 159 (1989)
[12] T Davis, *Introduction to nonlinear differential .and integral equations* (Dover Publ., New York, 1962)