

Energy and momentum of cylindrical gravitational waves-II

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Abstract. Recently Rosen and the present author obtained the energy and momentum densities of cylindrical gravitational waves in Einstein's prescription and found them to be finite and reasonable. In the present paper we calculate the same in the prescriptions of Tolman as well as Landau and Lifshitz and discuss the results.

Keywords. Einstein–Rosen metric; energy-momentum pseudotensors.

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1. Introduction

Long ago Scheidegger [1] raised doubts whether gravitational radiation has well-defined existence. To this end, Rosen [2] investigated whether or not the cylindrical gravitational waves have energy and momentum. He used the energy-momentum pseudotensors of Einstein and Landau and Lifshitz (LL) and carried out calculations in cylindrical polar coordinates and found that the energy and momentum density components vanish. The results obtained by him fit in with the conjecture of Scheidegger that a physical system cannot radiate gravitational energy. Two years later, Rosen [3] realized the mistake and carried out the calculations in “cartesian coordinates” and reported that the energy and momentum density are nonvanishing and reasonable. Though in the letter he remarked that he would publish the details elsewhere, it was not done until recently he and the present author [4] re-calculated the energy and momentum density components in “cartesian coordinates” in Einstein's prescription. They found finite and reasonable results.

The physical interpretation of Einstein's energy-momentum complex has been questioned by several physicists, notably by Weyl, Pauli and Eddington [5]. Their objections were that the energy-momentum complex of Einstein was neither a tensor nor was it symmetric. However, Palmer [6] discussed the importance of Einstein's energy-momentum complex in detail. Since the first energy-momentum complex of Einstein was given, a large number of prescriptions to obtain energy and momentum in a general relativistic system have been proposed by many authors [7]. Since there is no unique way of defining energy and momentum in a curved spacetime, and also the various energy-momentum complexes are not tensors, some researchers do not take them seriously. They even guess that the different energy-momentum complexes could give different and therefore unacceptable energy distribution in a given spacetime. However, we have shown that the energy-momentum pseudotensors of Einstein, Tolman and LL give the same result for energy distribution in the Kerr–Newman as

well as in the Vaidya spacetimes when calculations are carried out in “cartesian coordinates” [8–10]. We have already mentioned that Rosen and the present author obtained energy and momentum density for the cylindrical gravitational waves in “cartesian coordinates” in Einstein’s prescription and found reasonable result. Therefore, it is of interest to investigate whether or not other energy-momentum complexes give the same result. This is the aim of this paper. We consider here the energy-momentum complexes of Tolman and LL. This paper is organized as follows. Section 2 gives the Einstein–Rosen metric describing the cylindrical gravitational waves. In §3 and §4, we obtain energy and momentum density of cylindrical gravitational waves in prescriptions of Tolman and LL. In §5 we discuss the results obtained. We use the geometrized units such that $G = 1, c = 1$ and follow the convention that the Latin and Greek indices take values 0 to 3 and 1 to 3, respectively. x^0 is the time coordinate.

2. Einstein–Rosen metric

The Einstein–Rosen metric is a non-static vacuum solution of Einstein’s field equations and it describes the gravitational field of cylindrical gravitational waves. It is given by the line element [11]:

$$ds^2 = e^{2\gamma - 2\psi}(dt^2 - d\rho^2) - e^{-2\psi}\rho^2 d\phi^2 - e^{2\psi} dz^2, \quad (1)$$

where $\gamma = \gamma(\rho, t), \psi = \psi(\rho, t)$, and

$$\psi_{\rho\rho} + \frac{\psi_\rho}{\rho} - \psi_{tt} = 0, \quad (2)$$

$$\gamma_\rho = \rho(\psi_\rho^2 + \psi_t^2), \quad (3)$$

$$\gamma_t = 2\rho\psi_\rho\psi_t. \quad (4)$$

The subscripts on ψ and γ denote partial derivatives with respect to the subscripts. It is known that the energy-momentum complexes of Tolman and LL, like that of Einstein, give correct result if calculations are carried out in “cartesian coordinates” [12–17]. Therefore, one transforms the line element, given by (1), according to

$$\begin{aligned} x &= \rho \cos \phi, \\ y &= \rho \sin \phi, \end{aligned} \quad (5)$$

and gets the line element in t, x, y, z coordinates [4],

$$ds^2 = e^{2(\gamma - \psi)} \left[dt^2 - \frac{(x dx + y dy)^2}{\rho^2} \right] - \frac{e^{-2\psi}}{\rho^2} (x dy - y dx)^2 - e^{2\psi} dz^2. \quad (6)$$

3. Energy and momentum of cylindrical gravitational waves in prescription of Tolman

The energy-momentum complex of Tolman is [13–14]

$$\mathcal{F}_k^i = \frac{1}{8\pi} U_{k,j}^{ij}, \quad (7)$$

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where

$$U_k^{ij} = \sqrt{-g} \left[-g^{ii} (-\Gamma_{kl}^j + \frac{1}{2}g_k^j \Gamma_{al}^a + \frac{1}{2}g_l^j \Gamma_{ak}^a) + \frac{1}{2}g_k^i g^{lm} (-\Gamma_{lm}^j + \frac{1}{2}g_l^j \Gamma_{am}^a + \frac{1}{2}g_m^j \Gamma_{al}^a) \right]. \quad (8)$$

\mathcal{F}_0^0 is the energy density, \mathcal{F}_α^0 are the momentum density components, and \mathcal{F}_0^α are the components of energy current density. The required nonvanishing components of U_k^{ij} are

$$\begin{aligned} U_0^{01} &= \frac{x(e^{2\gamma} - 1)}{2\rho^2}, \\ U_0^{02} &= \frac{y(e^{2\gamma} - 1)}{2\rho^2}, \\ U_1^{01} &= \frac{-2\psi_t \psi_\rho y^2}{\rho}, \\ U_2^{02} &= \frac{-2\psi_t \psi_\rho x^2}{\rho}, \\ U_1^{02} &= U_2^{01} = \frac{2\psi_t \psi_\rho xy}{\rho}, \\ U_3^{03} &= 2\psi_t (1 - \rho\psi_\rho), \\ U_0^{11} &= \frac{2e^{2\gamma} \psi_t \psi_\rho y^2}{\rho}, \\ U_0^{22} &= \frac{2e^{2\gamma} \psi_t \psi_\rho x^2}{\rho}, \\ U_0^{12} &= U_0^{21} = \frac{-2e^{2\gamma} \psi_t \psi_\rho xy}{\rho}, \\ U_0^{33} &= 2e^{2\gamma} \psi_t (\rho\psi_\rho - 1). \end{aligned} \quad (9)$$

Using (9) in (7) we obtain

$$\begin{aligned} \mathcal{F}_0^0 &= \frac{e^{2\gamma}(\psi_\rho^2 + \psi_t^2)}{8\pi}, \\ \mathcal{F}_1^0 &= \frac{x\psi_\rho \psi_t}{4\pi\rho}, \\ \mathcal{F}_2^0 &= \frac{y\psi_\rho \psi_t}{4\pi\rho}, \\ \mathcal{F}_0^1 &= -e^{2\gamma} \mathcal{F}_1^0, \\ \mathcal{F}_0^2 &= -e^{2\gamma} \mathcal{F}_2^0, \\ \mathcal{F}_0^3 &= \mathcal{F}_3^0 = 0. \end{aligned} \quad (10)$$

4. Energy and momentum of cylindrical gravitational waves in prescription of LL

The energy-momentum complex of LL is [15]

$$L^{mn} = \frac{1}{16\pi} S^{mjnk}{}_{,jk}, \quad (11)$$

where

$$S^{mjnk} = -g(g^{mn}g^{jk} - g^{mk}g^{jn}). \quad (12)$$

L^{mn} is symmetric in its indices. L^{00} is the energy density and $L^{0\alpha}$ are the momentum (energy current) density components. S^{mjnk} has symmetries of the Riemann curvature tensor. The required nonvanishing components of S^{mjnk} are

$$\begin{aligned} S^{0101} &= \frac{-(e^{2\gamma}y^2 + x^2)}{\rho^2}, \\ S^{0102} &= \frac{xy(e^{2\gamma} - 1)}{\rho^2}, \\ S^{0202} &= \frac{-(e^{2\gamma}x^2 + y^2)}{\rho^2}, \\ S^{0303} &= -e^{2(\gamma-2\psi)}. \end{aligned} \quad (13)$$

Making use of (13) in (11) we obtain energy and momentum (energy current) density components for the cylindrical gravitational waves in LL prescription

$$\begin{aligned} L^{00} &= \frac{e^{2\gamma}(\psi_\rho^2 + \psi_t^2)}{8\pi}, \\ L^{01} &= \frac{-e^{2\gamma}\psi_\rho\psi_t x}{4\pi\rho}, \\ L^{02} &= \frac{-e^{2\gamma}\psi_\rho\psi_t y}{4\pi\rho}, \\ L^{03} &= 0. \end{aligned} \quad (14)$$

5. Discussion

One can see from (10) and (14) that the energy-momentum complexes of Tolman and LL give the same energy and energy current densities as given by Einstein's prescription [4]. The momentum density components in the Tolman prescription are the same as in the Einstein prescription. The momentum density components in Tolman and LL prescriptions differ by a sign, as expected. The energy density of the cylindrical gravitational waves is finite and positive definite, and the momentum density components reflect the symmetry of the spacetime. Thus our investigations do not support the conjecture of Scheidegger [1].

Due to non-tensorial nature of energy-momentum complexes, some physicists do not take them seriously. They even conjecture that different energy-momentum complexes could give different energy distributions in a given spacetime. To this end we

have shown that the well-known energy-momentum complexes of Einstein, Tolman and LL give the same and reasonable result for the Kerr–Newman, Vaidya, and Einstein–Rosen spacetimes (see also ref. 4, 7–10).

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Note added in proof:

Using the Papapetrou pseudotensor we have obtained the energy and energy current (momentum) densities of the cylindrical gravitational waves (described by the Einstein–Rosen metric) and have found them to be the same as given by the LL energy momentum complex.