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Static models of cosmic strings in general relativity

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Abstract. Exact solutions to Einstein's equations for a cloud of massive strings with a general static metric representing spherical plane and hyperbolic symmetries are derived. Some properties of massive strings for different cases are also discussed.

Keywords. Cosmic strings; static models; cloud of massive strings.

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1. Introduction

Cosmic strings have received considerable attention as they are believed to have served in the structure formation in the early stages of the universe. Cosmic strings may have been created during phase transitions in the early universe [1] and they could act as a source of gravitational field [2]. The existence of strings in the early universe can be used to introduce density fluctuations leading to the formation of galaxies [3–5]. A gauge-invariant model of a cloud of geometric strings with spherical, plane and cylindrical symmetries has been studied by Letelier [6]. He has suggested that the massive strings are being formed by geometric strings with particle attached to them along their extension. The general relativistic treatment of strings has been originally given by Letelier [6] and Stachel [7]. Relativistic models of cosmic strings in Bianchi type II, VI_0, VIII and IX space times have been studied by Krori *et al* [8]. Banerjee *et al* [10] have obtained Bianchi type I string cosmological models with magnetic field. Chakraborty [9] and Tikekar and Patel [11] have studied different cases of Bianchi type VI_0 string cosmology with and without magnetic field. Recently, Tikekar *et al* [12] have presented a new class of singularity free cylindrically symmetric models in string cosmology.

The study of static models of the universe is appealing not only for theoretical reasons, but also because, if there exists an explanation for the observed red shift other than the expansion of the universe, the static models could gain importance [13]. Gott [14] and Linet [15] had considered static cylindrically symmetric interior and exterior space-times of an extended cosmic strings with uniform and non-uniform energy densities. The metric of the static space-time describing infinite straight cosmic strings of linear mass density has been obtained by Letelier [16]. Recently, Krori *et al* have studied the field of a stationary cosmic string.

In this paper we are considering a more general static metric with spherical, plane and hyperbolic symmetries to study a cloud of massive strings in different cases.

2. Field equations

We consider the most general static line element for matter distribution given by

$$ds^2 = g^2(x)dt^2 - dx^2 - f^2(x)[dy^2 + F(y)dz^2] \tag{1}$$

which represents spherical $[F(y) = \sin^2 y]$, plane $[F(y) = 1]$ or hyperbolic $[F(y) = e^{2y}]$ symmetry.

The non-zero components of the Ricci curvature tensor are

$$\begin{aligned} R_{00} &= gg'' + 2\left(\frac{g}{f}\right)g'f' \\ R_{11} &= \frac{g''}{g} - \frac{2f''}{f} \\ R_{22} &= \frac{R_{33}}{F} = \varepsilon(F) - (f')^2 - ff'' - \left(\frac{f}{g}\right)g'f' \end{aligned} \tag{2}$$

with

$$\varepsilon(F) = \frac{\dot{F}}{2F} - \frac{\ddot{F}}{2F} = \begin{cases} 1 & \text{for } F = \sin^2 y \\ 0 & \text{for } F = 1 \\ -1 & \text{for } F = e^{2y} \end{cases}$$

and Ricci scalar

$$R = -\frac{2}{f^2}\varepsilon(F) + 2\frac{g''}{g} + 4\frac{f''}{f} + 2\left(\frac{f'}{f}\right)^2 + 4\frac{f'g'}{fg}. \tag{3}$$

The Einstein's equation for a cloud of strings is

$$R_{ij} - \tfrac{1}{2}g_{ij}R = -8\pi T_{ij} \tag{4}$$

where the energy-momentum tensor of a cloud of massive strings has been given by

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j \tag{5}$$

with

$$u^i u_i = -x^i x_i = 1, \quad u^i x_i = 0.$$

Here ρ denotes the rest energy density of the strings with particles attached to them (P-strings) and λ is the string tension density. If we denote particle energy-density by ρ_p, then we have

$$\rho = \rho_p + \lambda. \tag{6}$$

The vector u^i describes the cloud four-velocity and x^i is a unit space-like vector representing the direction of the string.

If we assume that the space-like vector x^i is parallel to $\partial/\partial x$ (x^i must be taken along any one of the three directions $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$) so that the energy-momentum tensor has the form

$$T_0^0 = \rho; \quad T_x^x = \lambda. \tag{7}$$

For the metric (1), the field (4) with the help of (2), (3) and (7), may be written as

$$2\frac{f''}{f}+\left(\frac{f'}{f}\right)^2-\frac{\varepsilon}{f^2}=8\pi\rho \tag{8}$$

$$\left(\frac{f'}{f}\right)^2+2\frac{f'g'}{fg}-\frac{\varepsilon}{f^2}=8\pi\lambda \tag{9}$$

$$\frac{f''}{f}+\frac{g''}{g}+\frac{f'g'}{fg}=0 \tag{10}$$

By combining (8)–(10), we have

$$\lambda'+(\lambda-\rho)\frac{g'}{g}+2\lambda\frac{f'}{f}=0. \tag{11}$$

3. Solutions of the equations

To solve the Einstein's equations we require one more relation between the variables because there are four variables in three independent equations.

Case I: Nambu strings

Consider the equation of state for a cloud of strings as

$$\rho=\lambda \quad (\rho_p=0) \tag{12}$$

then (11) gives the solution

$$\lambda=\frac{D}{f^2} \tag{13}$$

where D is a constant.
Using (12) and (13) in (8) and multiplying by f^2f' and integrating over x, we have

$$f=(8\pi D+\varepsilon)^{1/2}x+E \tag{14}$$

where E is an integration constant.

Substracting (9) from (8) and using (12), and then integrating over x, we get

$$g=K \text{ (constant)}. \tag{15}$$

From (12)–(14) we see that the rest energy density of the cloud of strings and the string tension density are decreasing as the field is moving along the x-axis in all the three cases viz., spherical, plane and hyperbolic symmetries.

Case II.

In this case we consider a relation between the metric coefficients as given by

$$f=g^n \tag{16}$$

where n is an arbitrary constant.

Substituting this value of f in (10), we get

$$(n+1)\frac{g''}{g}+n^2\left(\frac{g'}{g}\right)^2=0 \tag{17}$$

for which the solution is

$$g=[Ax+B]^{(n+1)/(n^2+n+1)} \tag{18}$$

where A and B are arbitrary constants.

Using the value of f and g from (16) and (18) in (8) and (9), we have

$$8\pi\rho=\frac{n(n^2-1)(n+2)A^2}{(n^2+n+1)^2(Ax+B)^2}-\frac{\varepsilon}{(Ax+B)^{2n(n+1)/(n^2+n+1)}} \tag{19}$$

$$8\pi\lambda=\frac{n(n+1)^2(n+2)A^2}{(n^2+n+1)^2(Ax+B)^2}-\frac{\varepsilon}{(Ax+B)^{2n(n+1)/(n^2+n+1)}}. \tag{20}$$

From (6), (19) and (20) we get

$$8\pi\rho_{\mathrm{p}}=-2\frac{n(n+1)(n^2+n-2)A^2}{(n^2+n+1)^2(Ax+B)^2}. \tag{21}$$

Substituting the value of $\varepsilon=0,1,-1$ we can obtain the solutions of massive cosmic strings in plane, spherical and hyperbolic symmetric space-times respectively.

From (19)–(21) one can see that the rest energy density, tension density and particle density of the strings are decreasing as the field is moving along the x-axis in all the three symmetries and all the three densities vanish when $x\to\infty$.

Case III: Uniform density

If we consider the rest energy density of the cloud of strings to be uniform i.e., $\rho=\rho_0$ (constant), then (8) reduces to

$$2\frac{f''}{f}+\left(\frac{f'}{f}\right)^2-\frac{\varepsilon}{f^2}=8\pi\rho_0. \tag{22}$$

Multiplying (22) by f^2f' and integrating, we obtain

$$f'=\left(\frac{8\pi\rho_0}{3}f^2+\varepsilon\right)^{1/2} \tag{23}$$

Now, we consider a plane symmetric space-time ($\varepsilon=0$). Substituting $\varepsilon=0$ in (23) yields the solution

$$f=b\exp\left(\left(\frac{8\pi\rho_0}{3}\right)^{1/2}x\right) \tag{24}$$

where b is an integration constant.

Using (24) in (10), one can obtain

$$g''+(8\pi\rho_0/3)^{1/2}g'+(8\pi\rho_0/3)g=0 \tag{25}$$

which has the solution

$$g = (d\cos(2\pi\rho_0)^{1/2}x) + c\sin((2\pi\rho_0)^{1/2}x))\exp[-(2\pi\rho_0)^{1/2}x] \tag{26}$$

where c and d are arbitrary constants.

From (9), (24) and (26), we have

$$\lambda = \frac{8\pi\rho_0}{(3)^{1/2}}\frac{(-d\sin((2\pi\rho_0)^{1/2}x) + c\cos((2\pi\rho_0)^{1/2}x))}{(d\cos((2\pi\rho_0)^{1/2}x) + c\sin((2\pi\rho_0)^{1/2}x))}. \tag{27}$$

Equations (6) and (27) give

$$\rho_p = \rho_0 - \frac{8\pi\rho_0}{(3)^{1/2}}\frac{(-d\sin((2\pi\rho_0)^{1/2}x) + c\cos((2\pi\rho_0)^{1/2}x))}{(d\cos((2\pi\rho_0)^{1/2}x) + c\sin((2\pi\rho_0)^{1/2}x))}. \tag{28}$$

From (27) it is clear that when x approaches $(2\pi\rho_0)^{-1/2}\tan^{-1}\left(\frac{c}{d}\right)$ the string tension density vanishes and the cloud of particles would attain uniform energy density. Thus in this model a cloud of strings with particles attached to them asymptotically tends to a cloud of particles.

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