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The Coulomb Green's function revisited

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Abstract. It is demonstrated how the energy-dependent Green's function for the Schrödinger–Coulomb problem can be deduced from a knowledge of the harmonic oscillator time-propagator. All the known results of the Coulomb system are shown to be elegantly derivable from such a connection.

Keywords. Coulomb-oscillator connection; Coulomb energy-Green's function.

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1. Introduction

The purpose of this article is to show how the quantum mechanical Green's function for the Coulomb potential can be obtained from a knowledge of the Feynman propagator for the harmonic oscillator. The key ingredient in the analysis is a remarkably simple relation between the radial parts of the Green's functions for the oscillator and the Coulomb problems. This is an example of a general relation between the Green's functions for different potentials [1]. That there exists a connection between the oscillator and Coulomb systems is something known for a long time. What is more recent is the realization that this connection takes the most succinct and useful form when expressed in terms of the radial Green's function. It enables one to extract all the results pertaining to one physical system when the Green's function for a related system is known.

It is somewhat surprising that the Green's function for the Schrödinger equation with the Coulomb potential was not known completely until the sixties, given the fact that the exact treatment of the hydrogen atom was one of the early triumphs of quantum mechanics. The first general expression for the Coulomb Green's function appears to be a double integral representation due to Wichmann and Woo [2]. A one-parameter integral for the Green's function in momentum space was derived by Schwinger [3]. It was Hostler [4, 5] who obtained for the first time a closed form expression in coordinate space. Starting with the bilinear expansion of the Green's function in terms of energy eigenfunctions, he carried out the summation over the bound states and the integration over the continuum, and expressed the resulting one-parameter integral as a sum of products of Whittaker functions of different arguments. More recently, several authors [6–8] have computed the Coulomb Green's function in the context of solving the hydrogen atom problem by path integration.

The starting point of our method is the time-dependent propagator for the three-dimensional isotropic harmonic oscillator. By Fourier-transforming the radial part of the propagator, we derive first an expression for the radial part of the Green's function.

Invoking the relation referred to earlier, we go from the oscillator radial Green's function to the Coulomb radial Green's function. Once the latter is known, all the results of the Coulomb problem can be deduced: the discrete spectrum and the normalized eigenfunctions, the continuum and its δ-normalized eigenfunctions, as well as the completeness property of the eigenfunctions. By attaching spherical harmonics to the radial Green's function appropriately, and summing the resulting partial wave series, we obtain an expression for the complete Coulomb Green's function as an integral over a single variable.

The main advantage of our method is that there is no need to deal with the continuum and discrete states of the Coulomb potential separately. This is because we consider the partial wave expansion of the Green's function, rather than its eigenfunction expansion, and focus attention on the radial part which carries all the essential information, and contains so to speak both the discrete and continuum states. The angular degrees of freedom can thus be decoupled at the beginning and brought back at the end. Further, our method is simple and transparent, and leads to the final expression for the Green's function with much less labour.

2. Radial Green's function for the harmonic oscillator

The time-dependent propagator for a particle of unit mass moving from a position $\mathbf{r}'$ at time t' to a position $\mathbf{r}$ at time t in a potential V satisfies the equation ($\hbar = 1$)

$$\left[i\frac{\partial}{\partial t} + \frac{1}{2}\nabla^2 - V\right]K(\mathbf{r}t, \mathbf{r}'t') = i\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{1}$$

If the potential is time-independent the propagator depends on time only through $T = t - t'$, and if V is spherically symmetric, then it has also the partial wave expansion

$$K(\mathbf{r}, \mathbf{r}', T) = \sum_l \sum_m \frac{K_l(r, r', T)}{rr'} Y^*_{lm}(\theta, \phi)\, Y_{lm}(\theta', \phi'). \tag{2}$$

The function $K_l(r, r', T)$ defined by (2) is the radial propagator. For the isotropic harmonic oscillator with $V = \frac{1}{2}\omega^2 r^2$, the radial propagator is known to be given by the expression [9]

$$K_l^{\rm osc}(r, r', T) = \frac{\omega\sqrt{rr'}(-i)^{l+3/2}}{\sin \omega T} \exp\left[\frac{1}{2} i\omega(r^2 + r'^2)\cot \omega T\right] J_{l+1/2}\left(\frac{\omega rr'}{\sin \omega T}\right), \tag{3}$$

where $J_{l+1/2}$ is the Bessel function of the first kind.

The Green's function $G(\mathbf{r}, \mathbf{r}', E)$ which is a function of the energy E is related to the propagator by a Fourier transformation. For a central potential the angular dependence of G can be separated out exactly as in (2), with the radial part $g_l(r, r', E)$ replacing K_l on the rhs. For g_l we adopt the following definition:

$$g_l(r, r', E) = i\int_0^\infty \mathrm{d}T \exp(iET) K_l(r, r', T). \tag{4}$$

In (4) E is to be understood as $E + i\delta$ whenever necessary, together with the limit $\delta \to 0+$.

Our objective now is to obtain an expression for the oscillator Green's function g_l^{osc} using (3) and (4). To this end we consider the integral [10]

$$I = \int_0^\infty \left(\coth\frac{x}{2}\right)^{2\lambda} \exp(-\beta\cosh x) J_{2\mu}(\alpha\sinh x)\mathrm{d}x. \tag{5}$$

This integral exists provided the constants α, β, λ and μ satisfy the following two conditions:

$$\mathrm{Re}(\mu - \lambda) > -\tfrac{1}{2}, \quad \mathrm{Re}\,\beta > |\mathrm{Im}\,\alpha|. \tag{6}$$

It has the value

$$I = \frac{\Gamma(\frac{1}{2} - \lambda + \mu)}{\alpha\Gamma(2\mu + 1)} M_{-\lambda,\mu}(\sqrt{\alpha^2 + \beta^2} - \beta)\, W_{\lambda,\mu}(\sqrt{\alpha^2 + \beta^2} + \beta) \tag{7}$$

where M and W are the Whittaker functions defined by the relations

$$M_{\lambda,\mu} = z^{(1/2)+\mu}\exp(-z/2)\Phi(\mu - \lambda + \tfrac{1}{2}, 2\mu + 1; z), \tag{8}$$

$$W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \lambda)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \lambda)} M_{\lambda,-\mu}(z). \tag{9}$$

In (8) $\Phi(a, c, z)$ is the confluent hypergeometric function which is regular at the origin.

We now change the variable in (5) by writing $\sinh x = \mathrm{cosech}\, t$. This transforms the integral into

$$I = \int_0^\infty \mathrm{d}t \exp(2\lambda t)\mathrm{cosech}\, t \exp[-\beta\coth t] J_{2\mu}(\alpha\,\mathrm{cosech}\, t). \tag{10}$$

The two conditions given in (6) ensure the convergence of the integral at the upper and lower limits respectively. Further, the first condition also ensures that the integrand in (10) [which we denote by $f(t)$] vanishes exponentially as $|t| \to \infty$ in the first quadrant of the complex t plane. Let us consider now a quarter circle of radius R in the first quadrant centred at the origin and bounded by the axes. Within this closed contour the function $f(t)$ has no singularities. Therefore an application of Cauchy's theorem yields, in the limit $R \to \infty$, the result

$$\int_0^\infty f(t)\mathrm{d}t = \int_0^{i\infty} f(t)\mathrm{d}t.$$

Using this in (10) and writing $t = i\tau$, we get

$$I = i\int_0^\infty \mathrm{d}\tau \exp(2i\lambda\tau)\mathrm{cosec}\,\tau \exp(i\beta\cot\tau) J_{2\mu}(-i\alpha\,\mathrm{cosec}\,\tau). \tag{11}$$

If we make the replacements

$$\tau = \omega T, \quad 2\lambda\omega = E, \quad \beta = \tfrac{1}{2}\omega(r^2 + r'^2), \quad \alpha = i\omega r r', \quad 2\mu = l + \tfrac{1}{2} \tag{12}$$

in (11), the result is

$$I = \omega\int_0^\infty \mathrm{d}T \frac{\exp(iET)}{\sin\omega T} \exp\left[\frac{1}{2} i\omega(r^2 + r'^2)\cot\omega T\right] J_{l+1/2}\left(\frac{\omega r r'}{\sin\omega T}\right). \tag{13}$$

It follows from (3), (4) and (13) that the oscillator radial Green's function is given by

$$g_l^{\rm osc}(r,r',E) = (-i)^{l+1/2}\sqrt{rr'}\,I. \tag{14}$$

Using (7) to replace I on the rhs of (14). we finally arrive at the explicit expression

$$g_l^{\rm osc}(r,r',E) = \frac{\Gamma\left(-\dfrac{E}{2\omega}+\dfrac{l}{2}+\dfrac{3}{4}\right)}{\omega\sqrt{rr'}\,\Gamma\left(l+\dfrac{3}{2}\right)} M_{\lambda,\mu}(\omega r'^2)\, W_{\lambda,\mu}(\omega r^2), \tag{15a}$$

$$\lambda = E/2\omega, \quad \mu = \tfrac{1}{2}(l+\tfrac{1}{2}). \tag{15b}$$

The above form holds for $r > r'$; for $r < r'$ the arguments of the two Whittaker functions must be interchanged. In obtaining (15), a phase factor $(-i)^{l+3/2}$ has been absorbed, by making use of the reflection property

$$(-z)^{-(1/2)-\mu} M_{-\lambda,\mu}(-z) = z^{-(1/2)-\mu} M_{\lambda,\mu}(z). \tag{16}$$

We note that whenever the argument of the gamma function in the numerator of (15) is a non-positive integer $-n_r$, $g_l^{\rm osc}$ has a pole in the energy plane at $E=(2n_r+l+3/2)\omega$. These are the allowed energies of the oscillator for a fixed value of l. There are no other singularities because $M_{\lambda,\mu}$ and $W_{\lambda,\mu}$ are entire functions of λ. This follows from the definitions (8) and (9) and the fact that $\Phi(a,c,z)$ is an entire function of a.

3. Coulomb radial Green's function

The energy-dependent Green's function associated with the Coulomb potential $V(r) = -e^2/r$ (hydrogen atom) can be determined from a knowledge of $g_l^{\rm osc}$. We have shown in an earlier work [1] that the radial part of the Coulomb Green's function is related in a remarkably simple manner to the oscillator radial Green's function. If we denote by $g_L^{\rm osc}(r,r',\varepsilon,\omega)$ the latter, corresponding to an oscillator of frequency ω (indicated explicitly now), energy ε and angular momentum L, the precise connection between the two is

$$g_l^{\rm Cou}(r,r',E) = 2(rr')^{1/4} g_{2l+1/2}^{\rm osc}(\sqrt{r},\sqrt{r'},\varepsilon=4e^2,\omega=\sqrt{-8E}). \tag{17}$$

In other words, the Coulomb radial Green's function for a given l and energy E is determined by the radial Green's function of an oscillator of frequency $\sqrt{-8E}$, energy $4e^2$ and angular momentum continued to the value $2l+1/2$. Inserting (15) into (17), we obtain the expression

$$g_l^{\rm Cou}(r,r',E) = \frac{2\Gamma(-\lambda+l+1)}{(2l+1)!\,\omega} M_{\lambda,\mu}(\omega r')\, W_{\lambda,\mu}(\omega r), \tag{18a}$$

with

$$\omega = \sqrt{-8E}, \quad \lambda = 2e^2/\omega, \quad \mu = l+\tfrac{1}{2}. \tag{18b}$$

Using (18) we can determine the complete Green's function for the Coulomb problem. Deferring this to a later section, we now consider how the Coulomb wave functions can be deduced from $g_l^{\rm Cou}$ which will be simply called g_l hereafter for convenience.

4. Energy eigenfunctions of the hydrogen atom

The singularities of the function g_l in the E plane constitute the spectrum of allowed energies of the hydrogen atom. We see from (17) that g_l has simple poles when

$$-2e^2/\omega + l + 1 = -n_r, \quad n_r = 0, 1, 2, \ldots$$

These correspond to bound states with energies (cf. 18b)

$$E = E_n = -e^4/2(n_r + l + 1)^2 = -e^4/2n^2, \tag{19}$$

where $n = n_r + l + 1$ is the principal quantum number.

The bound state wave functions are given by the residue of g_l at its poles. If we denote by u_{nl} the reduced radial wave function, we have

$$\mathrm{Res}[g_l(r, r', E)]_{E=E_n} = \lim_{E \to E_n} [(E_n - E)g_l] = u_{nl}(r)u_{nl}(r'). \tag{20}$$

At the pole $\omega = \omega_n = 2/na$, where $a = e^{-2}$ is the Bohr radius, and

$$u_{nl}(r)u_{nl}(r') = \lim_{\omega \to \omega_n} \left[\frac{1}{8}(\omega^2 - \omega_n^2) g_l \right]. \tag{21}$$

Also, $\lambda = -n$ in the limit; for this value of λ the Whittaker functions $M_{\lambda, l+1/2}$ and $W_{\lambda, l+1/2}$ are both expressible in terms of Laguerre polynomials in a standard way. A straightforward evaluation of the above limit yields the wave function

$$u_{nl}(r) = \left[\frac{(n-l-1)!}{n^2 a (n+l)!^3} \right]^{1/2} \left(\frac{2r}{na} \right)^{l+1} \exp(-r/na)\, L_{n+l}^{2l+1} \left(\frac{2r}{na} \right). \tag{22}$$

It may be noted that the wave function comes out with the correct normalization factor. The reason for this is to be traced ultimately to the normalization of the propagator K which is specified by the inhomogeneous eq. (1).

In addition to the poles g_l has a branch point at $E = 0$. To see this we must consider the behaviour of the Green's function near $\omega = 0$. First, we note that the Whittaker function $M_{\lambda, l+1/2}$ for $\lambda = 2e^2/\omega$ has the following behaviour as $\omega \to 0$:

$$M_{\lambda, l+1/2}(\omega r') \sim (\omega r')^{l+1} [A_0(r') + A_1(r')\omega + 0(\omega^2)].$$

This may be readily deduced from (8) on substituting the basic series representation of the confluent hypergeometric function Φ. Next we make use of an explicit series expansion of the Whittaker function $W_{\lambda,\mu}$ which is valid when $2\mu + 1$ is a positive integer [11]. This, coupled with the asymptotic expansion of $\Gamma(z + a)$, shows that for small ω

$$\Gamma(-\lambda + l + 1)\, W_{\lambda, l+1/2}(\omega r) \sim \omega^{-l} [B_0(r) + B_1(r)\omega + O(\omega^2)].$$

Hence the behaviour of the Green's function is given by

$$g_l(r, r', E) \sim C_0(r, r') + C_1(r, r')\omega + O(\omega^2). \tag{23}$$

This result implies that g_l has a square-root branch point in the energy plane at the origin. The associated branch cut can be taken to run along the positive real axis. This corresponds to choosing the principal value for the argument of E: $-\pi < \arg E \leqslant \pi$.

The occurrence of the branch point singularity is indicative of a continuum of positive energy eigenvalues for the Coulomb potential. The corresponding eigenfunctions are determined by the discontinuity of g_l across the branch out. Denoting by $v_l(r,E)$ the continuum eigenfunction belonging to some $E>0$, we have the relation

$$\begin{aligned}\operatorname{disc} g_l(r,r',E) &\equiv \lim_{\delta\to 0}\,[g_l(r,r',E+i\delta)-g_l(r,r',E-i\delta)]\\ &= 2\pi i v_l(r,E)v_l^*(r',E).\end{aligned} \tag{24}$$

Since for $E>0$, $\omega(E\pm io)=\mp i|\omega|$, we introduce the asymptotic wave number $k=\sqrt{2E}$ and write $\omega(E\pm io)=\mp 2ik$, $k>0$. Then the discontinuity can be written as

$$\begin{aligned}-ik(2l+1)!\operatorname{disc} g_l &= \Gamma(-\lambda'+\mu+\tfrac{1}{2})M_{\lambda',\mu}(-2ikr')\,W_{\lambda',\mu}(-2ikr)\\ &\quad+\Gamma(\lambda'+\mu+\tfrac{1}{2})M_{-\lambda',\mu}(2ikr')\,W_{-\lambda',\mu}(2ikr)\end{aligned} \tag{25}$$

where $\lambda'=ie^2/k$ and $\mu=l+1/2$. In order to identify the eigenfunction v_l, we must cast the rhs of (25) in a factorized form. To this end, we first replace the W functions in (25) in terms of the M's, using the relation (9). The resulting expression is rewritten by invoking the reflection property (16) of the M function, and then simplified by using the standard property $\Gamma(z)\Gamma(1-z)=\pi\operatorname{cosec}\pi z$ of the gamma function. The entire calculation is best done keeping μ arbitrary, and putting $\mu=l+1/2$ only at the end. The result is the product form

$$\begin{aligned}-ik(2l+1)!\operatorname{disc} g_l &= \frac{|\Gamma(\lambda'+l+1)|^2}{(2l+1)!}\exp(-i\pi\lambda')M_{\lambda,l+1/2}(-2ikr')\\ &\quad\times M_{-\lambda',l+1/2}(2ikr).\end{aligned} \tag{26}$$

From (26), (24) and (8), we get (apart from an over all phase factor) the continuum wave function:

$$\begin{aligned}v_l(r,E) &= \frac{\exp(\pi e^2/2k)}{\sqrt{2\pi k}(2l+1)!}\left|\Gamma\left(\frac{ie^2}{k}+l+1\right)\right|(2kr)^{l+1}\exp(-ikr)\\ &\quad\times\Phi\left(\frac{ie^2}{k}+l+1,2l+2,2ikr\right).\end{aligned} \tag{27}$$

This is the correct reduced radial wave function which is normalized according to

$$\int_0^\infty v_l^*(r,E)v_l(r,E)\mathrm{d}r=\delta(E-E'). \tag{28}$$

5. Completeness of the Coulomb eigenfunctions

The important property of completeness possessed by the Coulomb eigenfunctions can be deduced from the Green's function g_l as a direct consequence of its analytic properties in the energy plane. We present here an explicit demonstration.

For the present purpose it is more advantageous to consider g_l as a function of ω rather than of E. In the ω plane the only singularities are simple poles located on the positive real axis at $\omega = 2e^2/(n_r + l + 1)$, n_r being a non-negative integer. Unlike in the E plane, there is no singularity at $\omega = 0$, as shown by (23). As $\omega \to \infty$ in the half plane $\mathrm{Re}\,\omega > 0$, the Whittaker functions have the asymptotic behaviour

$$M_{\lambda,l+1/2} \to \frac{(2l+1)!}{l!} \exp\left(\frac{1}{2}\omega r'\right), \quad W_{\lambda,l+1/2} \to \exp\left(-\frac{1}{2}\omega r\right).$$

Therefore, the asymptotic form of g_l in the right half plane is

$$g_l \to \frac{2}{\omega} \exp\left[-\frac{1}{2}\omega(r - r')\right].$$

Consider now a contour in the ω plane made up of the imaginary axis from $\omega = iR$ to $\omega = -iR$ and a semicircle C_R lying in the right half plane with centre at the origin and radius $R > 2e^2/(l+1)$. All the poles of g_l lie within the contour. Integrating ωg_l around the contour, we have

$$\int_{C_R} \omega g_l(\omega) \mathrm{d}\omega + \int_{iR}^{-iR} \omega g_l(\omega) \mathrm{d}\omega = 2\pi i \sum (\text{residues of } \omega g_l). \tag{29}$$

The rhs can be immediately written down from (21):

$$\begin{aligned} \sum \text{residues} &= \sum_{n=l+1}^{\infty} [(\omega - \omega_n)\omega g_l(\omega)]_{\omega=\omega_n} \\ &= 4 \sum_{n=l+1}^{\infty} u_{nl}(r) u_{nl}(r'). \end{aligned} \tag{30}$$

Evaluating the lhs in the limit $R \to \infty$, the integral over the semicircle becomes

$$\begin{aligned} \int_{C_R} \omega g_l(\omega) \mathrm{d}\omega &\to \int_{C_R} 2 \exp\left[-\frac{1}{2}\omega(r - r')\right] \mathrm{d}\omega \\ &= \lim_{R \to \infty} 2 \int_{-iR}^{iR} \exp\left[-\frac{1}{2}\omega(r - r')\right] \mathrm{d}\omega \\ &= 8\pi i \delta(r - r'). \end{aligned} \tag{31}$$

As for the other integral on the lhs, we can write it as

$$\int_{iR}^{-iR} \omega g_l(\omega) \mathrm{d}\omega = \int_{iR}^{0} \omega g_l(i|\omega|) \mathrm{d}\omega + \int_{0}^{-iR} \omega g_l(-i|\omega|) \mathrm{d}\omega.$$

In view of the relation $\omega = \sqrt{-8E}$, we can identify, as was done earlier, $g_l(\pm i|\omega|) = g_l(E \mp io)$ with $E > 0$. Thus we have

$$\int_{iR}^{-iR} \omega g_l \mathrm{d}\omega = \int_{R^2/8}^{0} (-4\mathrm{d}E) g_l(E - io) + \int_{0}^{R^2/8} (-4\mathrm{d}E) g_l(E + io).$$

Now, using (24) and proceeding to the limit $R \to \infty$, we get

$$\int_{i\infty}^{-i\infty} \omega g_l(\omega) \mathrm{d}\omega = -8\pi i \int_{0}^{\infty} \mathrm{d}E\, v_l(r, E) v_l^*(r', E). \tag{32}$$

Putting (30), (31) and (32) in (29), we have the completeness relation

$$\sum_{n=l+1}^{\infty} u_{nl}(r)u_{nl}(r') + \int_0^{\infty} \mathrm{d}E\, v_l(r,E)v_l^*(r',E) = \delta(r-r'). \tag{33}$$

It may be noted that the above relation was also demonstrated by Mukunda [12] starting from the explicit expressions for the Coulomb wave functions.

6. The complete Green's function

The function g_l that we have considered so far is the radial part of the complete Green's function $G(\mathbf{r},\mathbf{r}',E)$, the connection between the two functions being given by the expansion

$$G(\mathbf{r},\mathbf{r}',E) = \sum_l \sum_m \frac{g_l(r,r',E)}{rr'} Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta',\phi'). \tag{34}$$

Making use of the relation

$$P_l(\cos\gamma) = \sum_m \frac{4\pi}{2l+1} Y_{lm}^*(\theta,\phi)\, Y_{lm}(\theta',\phi')$$

where γ is the angle between the two vectors $\mathbf{r}$ and $\mathbf{r}'$ and the explicit form (18) of g_l, we obtain the expression

$$G = \frac{1}{2\pi\omega rr'} \sum_l (2l+1)P_l(\cos\gamma)\frac{\Gamma(-\lambda+l+1)}{(2l+1)!} \times M_{\lambda,l+1/2}(wr')\, W_{\lambda,l+1/2}(\omega r). \tag{35}$$

Now the use of the formula [10]

$$\int_0^{\infty} \exp\left[-\frac{1}{2}(a+b)t\cosh x\right]\left(\coth\frac{x}{2}\right)^{2\lambda} I_{2\mu}(t\sqrt{ab}\sinh x)\mathrm{d}x = \frac{\Gamma(\frac{1}{2}+\mu-\lambda)}{t\sqrt{ab}\,\Gamma(1+2\mu)} W_{\lambda,\mu}(at)M_{\lambda,\mu}(bt)$$

which holds when $\mathrm{Re}(1/2+\mu-\lambda)>0$, $\mathrm{Re}\, t>0$ and $a>b>0$, enables us to rewrite (35) as

$$G = \frac{1}{2\pi\sqrt{rr'}} \sum_l (2l+1)P_l(\cos\gamma) \int_0^{\infty} \mathrm{d}x \exp\left[-\frac{1}{2}\omega(r+r')\cosh x\right] \times \left(\coth\frac{x}{2}\right)^{2\lambda} I_{2l+1}(\omega\sqrt{rr'}\sinh x). \tag{36}$$

Interchanging the order of the summation and the integration and applying the identity

$$\sum_l (2l+1)P_l(\cos\theta)I_{2l+1}(z) = \frac{1}{2}zI_0\left(z\cos\frac{\theta}{2}\right),$$

the summation in (36) can be performed. The result is

$$G=\frac{\omega}{4\pi}\int_0^\infty \mathrm{d}x\exp\left[-\frac{1}{2}\omega(r+r')\cosh x\right]\left(\coth\frac{x}{2}\right)^{2\lambda}\sinh x$$
$$\times I_0\left(\omega\sqrt{rr'}\cos\frac{\gamma}{2}\sinh x\right). \tag{37}$$

By the change of variable $\operatorname{cosech} t=\sinh x$, (37) can be transformed into

$$G=\frac{\omega}{4\pi}\int_0^\infty \mathrm{d}t\exp\left[-\frac{1}{2}\omega(r+r')\coth t+2\lambda t\right]\operatorname{cosech}^2 t$$
$$\times I_0\left(\omega\sqrt{rr'}\cos\frac{\gamma}{2}\operatorname{cosech} t\right). \tag{38}$$

In this expression, $\omega=\sqrt{-8E}$ and $\lambda=2e^2/\omega$.

Alternatively, we may proceed as follows. Starting with

$$G(\mathbf{r},\mathbf{r}',E)=\sum_l\frac{2l+1}{4\pi rr'}g_l(r,r',E)P_l(\cos\gamma)$$

we use (17), (14) and (13) to get

$$G=\frac{\omega}{2\pi\sqrt{rr'}}\sum_l(2l+1)P_l(\cos\gamma)(-i)^{2l+1}$$
$$\times\int_0^\infty\frac{\mathrm{d}T\exp(i4e^2T)}{\sin\omega T}\exp\left[\frac{1}{2}i\omega(r+r')\cot\omega T\right]J_{2l+1}\left(\frac{\omega\sqrt{rr'}}{\sin\omega T}\right). \tag{39}$$

Using now the identity

$$\sum_l(2l+1)P_l(-\cos\theta)J_{2l+1}(z)=\frac{1}{2}zJ_0\left(z\cos\frac{\theta}{2}\right),$$

we obtain a different representation

$$G=\frac{\omega^2}{4\pi i}\int_0^\infty\mathrm{d}T\exp\left[4ie^2T+\frac{1}{2}i\omega(r+r')\cot\omega T\right]$$
$$\times\operatorname{cosec}^2\omega T\,J_0\left(\omega\sqrt{rr'}\cos\frac{\gamma}{2}\operatorname{cosec}\omega T\right). \tag{40}$$

The above expression for the Coulomb Green's function is in agreement with those of Hostler [5] and Ho and Inomata [7] (modulo overall constant factors arising out of different definitions). At this point one can follow Hostler's steps to carry out the integration and express G in closed form in terms of Whittaker functions.

The logical end to the above analysis would be to compute the Coulomb propagator in closed form by Fourier-inverting the function G. This no one knows how to do as yet.

References

[1] S S Vasan, M Seetharaman and K Raghunathan, *Pramana – J. Phys.* **38**, 1 (1992)
[2] E H Wichmann and C-H Woo, *J. Math. Phys.* **2**, 178 (1961)
[3] J Schwinger, *J. Math. Phys.* **5**, 1606 (1964)
[4] L Hostler and R H Pratt, *Phys. Rev. Lett.* **10**, 469 (1963)
[5] L Hostler, *J. Math. Phys.* **5**, 591 (1964)
[6] H Duru and H Kleinert, *Phys. Lett.* **B84**, 185 (1979)
[7] R Ho and A Inomata, *Phys. Rev. Lett.* **48**, 231 (1982)
[8] F Steiner, *Phys. Lett.* **A106**, 363 (1984)
[9] D Peak and A Inomata, *J. Math. Phys.* **10**, 1422 (1969)
[10] I S Gradshteyn and I M Ryzhik, *Table of integrals, series and products* (Academic Press, New York. 1980) p. 729
[11] I S Gradshteyn and I M Ryzhik, *Table of integrals, series and products* (Academic Press, New York, 1980) p. 1063
[12] N Mukunda, *Am. J. Phys.* **46**, 910 (1978)