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Time dependent canonical perturbation theory I: General theory

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Abstract. In this communication, we reanalyze the causes of the singularities of canonical perturbation theory and show that some of these singularities can be removed by using time-dependent canonical perturbation theory. A study of the local and global properties (in terms of the perturbation parameter) is also undertaken.

Keywords. Canonical; perturbation; Hamiltonian systems; integrability.

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1. Introduction

For almost a hundred years, it has been known [1] that canonical perturbation theory is generically singular. Later works, most notably the KAM theorem [2] have shown that in spite of the singularities, the perturbation series does make sense in certain regions of phase space (the irrational tori), and hence, the orbits of the perturbed system lie close to those of the unperturbed system. In the case of the rational tori, even an infinitesimal perturbation is adequate to completely destroy the integrability and the orbits of the unperturbed system. Our analysis of perturbation theory, in this paper, tries to determine the exact cause of the singularities and shows how time-dependent perturbation theory can be used to study Hamiltonian systems which are close to integrable systems.

Consider the Hamiltonian

$$H = H_0(I) + \varepsilon H_1(I, \theta), \tag{1}$$

where I denote the action variables and θ denote the angle variables of the unperturbed Hamiltonian, H_0. (It is assumed that H_0 defines compact orbits.) In the language of Lie transforms, the aim of canonical perturbation theory is to determine a generator $\mathscr{F}(I, \theta)$ which intertwines between the time evolutions generated by H_0 and H:

$$\mathscr{F} \exp(-tH_0) = \exp(-tH)\mathscr{F} \tag{2}$$

as operators acting on phase space. To calculate $\mathscr{F}$, we write

$$\mathscr{F} = \ldots \exp(\varepsilon^n F_n) \ldots \exp(\varepsilon F_1) \tag{3}$$

where the action of the F_i is through Poisson Brackets:

$$\exp(\varepsilon^n F_n) f = f + \varepsilon^n \{F_n, f\} + \frac{\varepsilon^{2n}}{2} \{F_n, \{F_n, f\}\} + \ldots \tag{4}$$

for any function f on phase space. To order ε, and in the limit of infinitesimal t, we get

$$\exp(\varepsilon F_1) H_0 = H + O(\varepsilon^2), \tag{5}$$

matching terms of order ε leads us to the fundamental equation,

$$\{F_1, H_0\} = H_1. \tag{6}$$

We now recognize that the θ's represent angles; to make sense, all quantities, including H_1 and F_1 must be periodic in these angles. This motivates the Fourier expansions:

$$H_1 = \sum H_{1n}(I)\exp[i(n\cdot\theta)],$$
$$F_1 = \sum F_{1n}(I)\exp[i(n\cdot\theta)]; \tag{7}$$

Defining the frequencies of the unperturbed motion by $\omega = \partial H_0/\partial I$, we get the algebraic condition:

$$i(n\cdot\omega)F_{1n} = H_{1n}, \tag{8}$$

which leads to an obvious singularity whenever $(n\cdot\omega) = 0$ (the rational case).

1.1 *Reasons for singularities in the generator* F_1

It is worthwhile investigating at this stage the reason for this singularity. As we shall see, such an analysis will provide us with a way of circumventing the singularity.

A. Equation (6) can be considered as a partial differential equation for F_1. Using the method of characteristics, we see that the solutions of the characteristics are given by

$$\theta = \theta_0 + \omega t, \quad I = I_0, \tag{9}$$

where θ_0, I_0 represent the initial conditions. As can be seen, in the rational case, characteristics are closed; as a result, global solutions can be found only if suitable initial conditions can be formulated such that the value of F_1 on the initial surface matches with that obtained by integrating H_1 along the characteristic to the point where it reintersects the initial surface. This is possible iff the integral of H_1 along closed characteristics vanishes, i.e., iff $H_{1n} = 0$ whenever $(n\cdot\omega) = 0$. Thus, generically, the PDE has no global solutions.

B. Equivalently, (6) can be solved by using the formula

$$F_1 = \int \exp(-tH_0)H_1\,\mathrm{d}t, \tag{10}$$

where the integral is to be computed along the solutions to the equations of motion using H_0 as the Hamiltonian. In general, H_1 is periodic in time on the H_0 orbits and can contain a piece which is constant in time. Such a piece, on integration gives rise to a term which is linear in t and hence to a term which is linear in θ's (an aperiodic term). From this point of view, the singularity in F_1 arises from an attempt to express a term linear in angles as a periodic series.

C. Equation (6) can also be considered as an operator equation in a Hilbert space of periodic functions; such an equation admits solutions iff the RHS of the equation is orthogonal to the null eigenvectors of the adjoint of the operator, a condition that can easily be seen to be equivalent to the condition that $H_n = 0$ whenever $(n\cdot\omega) = 0$.

D. From a general point of view, any canonical transformation that is well-defined (e.g., analytic) and is time-independent is dynamically trivial: If $\mathscr{F}$ generates such a canonical transformation, then H_0 and H have the same spectrum: there is a 1–1 map between corresponding orbits. If H_0 and H have distinct dynamical contents (as would be expected to be the case), then the canonical transformation has to be either ill-defined (by being singular, as in canonical perturbation theory or by being multiple-valued) or time-dependent.

From this analysis, it is clear that the solution to opt for is to have either aperiodic terms in $\mathscr{F}$ or to allow for time-dependent $\mathscr{F}$'s. We shall assume the latter in this paper.

1.2 *Derivation of the time-dependent generating function*

To derive the equation for F_1, assume that g satisfies the equation

$$\frac{\partial g}{\partial t} + \{g, H_0\} = 0 \tag{11}$$

and define $h = \exp(\varepsilon F_1)g = g + \varepsilon\{F_1, g\} + O(\varepsilon^2)$. Using the condition that h be an invariant under H, i.e., $(\partial h/\partial t) + \{h, \mathrm{H}\} = 0$ gives the equation determining F_1:

$$\frac{\partial F_1}{\partial t} + \{F_1, H_0\} = H_1, \tag{12}$$

whose solution is

$$F_1(t) = \sum H_{1n} \exp[i(n\cdot\theta_0)] \frac{(\exp[i(n\cdot\omega)t] - 1)}{i(n\cdot\omega)} + F_{10} \tag{13}$$

where F_{10} is an arbitrary function of I and θ_0, representing the value on the initial surface $t = 0$, and where $\theta_0 = \theta - \omega t$. Thus,

$$F_1 = \sum H_{1n} \exp[i(n\cdot\theta)] \frac{(1 - \exp[-i(n\cdot\omega)t])}{i(n\cdot\omega)} + F_{10}, \tag{14}$$

where F_{10} is now an arbitrary function of I and $\theta - \omega t$.

The above solution can also be written in a different form which is sometimes more useful:

$$F_1(I, \theta, t) = \int_0^t H_1(I, \theta + \omega(z - t), z)\mathrm{d}z + F_{10} \tag{15}$$

which follows easily from the characteristic equations of the PDE for F_1 (eq. 12).

As is clear, F_{10} corresponds to the fact that we can choose the canonical transformation arbitrarily at $t = 0$. It is also easy to see that the time-dependent canonical transformation where F_{10} is zero is equivalent to using the unperturbed Hamiltonian to go back in time and to go forward in time using the perturbed Hamiltonian. In what follows, we will choose $F_{10} = 0$.

Another motivation for considering time-dependent perturbation theory comes from Hamilton–Jacobi theory. Usually, while applying Hamilton–Jacobi theory, one considers the generating function to be linear in time and writes the Hamilton–Jacobi equation in the form

$$H\left(\frac{\partial W}{\partial q}, q\right) = E \tag{16}$$

$$S = W - Et. \tag{17}$$

However, it is also possible to work with the full Hamilton–Jacobi equation: Assume,

$$H(I,\theta) = H_0(I) + \varepsilon H_1(I,\theta) \tag{18}$$

$$I = \frac{\partial S}{\partial \theta} \tag{19}$$

$$S(\theta,t) = S_0(\theta,t) + \varepsilon S_1(\theta,t) + \varepsilon^2 S_2(\theta,t) + \dots \tag{20}$$

Using these equations in the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H(I,\theta) = 0 \tag{21}$$

and equating equal powers of ε we get a series of equations determining S_i. It is then trivial to see that by allowing S to be explicitly dependent on time, we can derive an expression for S which is equivalent to the above Lie transform formalism.

To analyze the effect of such an F_1 and to draw conclusions about the integrability of the transformed system, it is convenient to replace the given dynamical system with one with an extra degree of freedom; the two extra phase space coordinates will be (t, T), and the new Poisson bracket is defined as,

$$\{f,g\}' = \frac{\partial f}{\partial t}\frac{\partial g}{\partial T} - \frac{\partial f}{\partial T}\frac{\partial g}{\partial t} + \{f,g\} \tag{22}$$

where the unprimed bracket refers to the original Poisson bracket. We also define

$$H'_0 = H_0 + T, \quad H' = H + T \tag{23}$$

and use z to denote the (new) arc parameter. We notice that there is no change in the definition of H_1. It is then trivial to see that the equation determining F_1 is the same as (6), using primed PBs, and that the solution coincides with that given in (15). We can thus interpret F_1 as a genuine function on the extended phase space defined by (I, T, θ, t). This also allows us to establish the integrability of H' (to order ε): we have $n+1$ invariants (I, T) for H'_0 and their transforms under the canonical transformation define invariants for H'. It is also straightforward to derive the equations for higher order perturbation theory (i.e., for $F_n, n > 1$) in a similar fashion.

2. Analyticity properties of the perturbation series

We now consider the analyticity properties of the perturbation series that we have developed above. There are two aspects to this study: first, we consider the finiteness of each term in the perturbation theory, that is, we consider the behaviour of each of the generators F_i as a function on phase space. Secondly, we consider the convergence of the canonical transformation as a whole, $\mathscr{F}$, as a function of the perturbation parameter.

2.1 *Analyticity properties of F_i*

We note that the function F_1 has the property of being an entire function of the two variables t and $(n\cdot\omega)$ (and hence I, if ω's and H_n are entire functions of I). This important property persists to all orders of perturbation theory:

Lemma. Assume that H'_0 and H_1 are analytic in I. Define a sequence of canonical transformations through the equations:

$$H^{(0)} = H'_0,$$

$$\{F_1, H'_0\}' = H_1, \tag{24}$$

$$H^{(n+1)} = \exp(\varepsilon^{(n+1)} F_{n+1}) H^{(n)}, \quad n \geqslant 0,$$

$$-\{F_{n+1}, H'_0\}' = \text{coefficient of } \varepsilon^{n+1} \text{ in } H^{(n)}, \quad n \geqslant 1. \tag{25}$$

(Note that at the nth stage, $F_{(n)}$ is chosen to kill the term of order ε^n.) Then, F_n is an entire function of I and t and is periodic in θ for all $n = 1, 2, \ldots$ (In brief, we shall say that F_n is regular.)

Proof. The proof follows from induction, using the following facts:

(1) If $H^{(n)}$ is regular and F_n is regular, so is $H^{(n+1)}$: this follows from the fact that the computation of $H^{(n+1)}$ involves the computation of derivatives, which preserves regularity.

(2) The PDE $\{A, H'_0\}' = G$ has the property that if G is regular, then A is regular; in fact, if $G = \Sigma G_n \exp[i(n\cdot\theta)]$, where the Fourier coefficients are entire functions of I and t, then $A = \Sigma A_n \exp[i(n\cdot\theta)]$, where

$$A_n = \exp[-i(n\cdot\omega)t] \int_0^t \exp[i(n\cdot\omega)t]\, G_n \mathrm{d}t. \tag{26}$$

The RHS is an integral of an entire function of t and $(n\cdot\omega)$ along a contour which lies within the region of analyticity; hence the conclusion.

Thus, in contrast to canonical perturbation theory, each of the generators is finite. At this stage it is worthwhile examining points (A) and (B) once again from the new perspective.

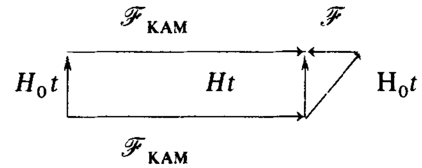
A′. In the extended system, the characteristics are never closed (the equation for t is $t = t_0 + z$); thus, it is possible to choose as initial surface, the surface $t = 0$ and choose F arbitrarily on this surface.

B′. Constant terms under the integral are retained explicitly as functions of time, not replaced by (aperiodic) functions of θ.

2.2 *Convergence of perturbation theory*

As noted above, the full canonical transformation $\mathscr{F}$ is equivalent to transforming backwards in time using H_0 and then transforming forwards in time using H. Since the first step is independent of ε, the analyticity properties of $\mathscr{F}$ is decided by the analyticity properties of the time evolution operator H as a function of ε. In general, of course, the analyticity properties of $\mathscr{F}$ as a function of ε will depend on t; this is in contrast to the time-independent case, where analyticity properties are independent of t. In general, it is not possible to derive general results regarding the analyticity properties of the transformation; however, there are two special results that are of interest:

1. We assume that there exists a time-independent canonical transformation $\mathscr{F}_{\mathrm{KAM}}$ (calculated using e.g., KAM theory) for a phase space point and for a certain value of ε. The following diagram shows the relation of $\mathscr{F}$ with the KAM generator.



which translates into

$$\mathscr{F} = \mathscr{F}_{\mathrm{KAM}} \exp(-tH_0) \mathscr{F}_{\mathrm{KAM}}^{-1} \exp(tH_0). \tag{27}$$

Thus, $\mathscr{F}$ will exist provided the time evolution of $\mathscr{F}_{\mathrm{KAM}}$ under the unperturbed Hamiltonian is well-defined for real t. In particular, since $\mathscr{F}_{\mathrm{KAM}}$ will depend on I, θ and their evolution under H_0 is trivial, $\mathscr{F}$ will be analytic in ε provided $\mathscr{F}_{\mathrm{KAM}}$ is analytic in I, θ. Further, in such a case, $\mathscr{F}$ will be analytic in ε for all t.

2. For a class of Hamiltonian systems it is possible to relate the complex-time analytical structure of fully perturbed Hamiltonian systems to the complex-ε analytical structure of the canonical transformation which transforms the Hamiltonian with a small ε value to the Hamiltonian with a large ε value.

Take a Hamiltonian system where the unperturbed part is a homogeneous function of degree m in phase space variables (q_i, p_i) and the perturbation part is homogeneous function of degree n.

$$H(q_i, p_i) = H_0(q_i, p_i) + \varepsilon H_1(q_i, p_i); \tag{28}$$

assuming $n > m$, a scale transformation of the variables,

$$q_i' = q_i \varepsilon^{\alpha}, \quad p_i' = p_i \varepsilon^{\alpha} \tag{29}$$

where $\alpha = (1/(n-m))$, changes the Hamiltonian into,

$$H'(q_i', p_i') = \varepsilon^{-m\alpha}(H_0(q_i', p_i') + H_1(q_i', p_i')) \tag{30}$$

(Note that this transformation is not a canonical transformation, but it transforms the vector field of H at some ε to the vector field of H at $\varepsilon = 1$.) Reparametrizing the system by inserting $\varepsilon = (1+\varepsilon')^{(1/m\alpha)}$ yields,

$$H'(q_i', p_i') = (1+\varepsilon')(H_0(q_i', p_i') + H_1(q_i', p_i')). \tag{31}$$

This Hamiltonian represents the perturbation of the Hamiltonian $H_{\varepsilon=1}(q_i', p_i') = H_0(q_i', p_i') + H_1(q_i', p_i')$ by itself using the perturbation parameter ε'. i.e.

$$H'(q_i', p_i') = H_{\varepsilon=1}(q_i', p_i') + \varepsilon' H_{\varepsilon=1}(q_i', p_i'). \tag{32}$$

For this problem, $F_1 = H_{\varepsilon=1}(q_i', p_i')t$, and all the higher order generators vanish; thus F_1 alone is the generator of the canonical transformation and so it can be used as the Hamiltonian for the ε' evolution (except at $\varepsilon = 0$, because there the scaling transformation is ill-defined).

$$\frac{\partial q_i'}{\partial \varepsilon'} = \frac{\partial F_1}{\partial p_i'}; \quad \frac{\partial p_i'}{\partial \varepsilon'} = -\frac{\partial F_1}{\partial q_i'}. \tag{33}$$

Now F_1 is the same as $H(q_i, p_i)$ at $\varepsilon = 1$, but for an extra multiplier t. So above equations can be rewritten as,

$$\frac{\partial q_i'}{\partial \varepsilon' t} = \frac{\partial H_{\varepsilon=1}(q_i', p_i')}{\partial p_i}; \quad \frac{\partial p_i'}{\partial \varepsilon' t} = -\frac{\partial H_{\varepsilon=1}(q_i', p_i')}{\partial q_i}. \tag{34}$$

But these equations are the time evolution equations for the Hamiltonian $H_{\varepsilon=1}$. Thus if $H_{\varepsilon=1}$ has singularities in the complex t plane, then a canonical transformation defined by F_1 also will have singularities in the complex $\varepsilon' t$ plane, and hence in the complex ε plane for a fixed t. Also, the presence of a natural boundary in the complex t plane [3] for the Hamiltonian $H_{\varepsilon=1}$ will manifest itself as a natural boundary in the complex ε plane for $\mathscr{F}$. (A similar technique will work for $n < m$.)

In case the singularities of the time evolution operator $\exp(-tH)$ in the complex ε plane are isolated, it is possible to use analytic continuation to define F beyond the radius of convergence. However, as the above study shows, the existence of natural boundaries in the complex t plane may imply the existence of similar boundaries in the complex ε plane also. In applications to specific systems, it is possible to study the analyticity properties of F using standard tools for determining the analyticity properties of the solutions of the equations of motion using H as the Hamiltonian.

3. Conclusions

The following points can be noted regarding the canonical transformations derived above:

1. Because of the explicit time-dependence of $\mathscr{F}$, two points which lie on the same unperturbed orbit do not in general lie on the same perturbed orbit.
2. The spectra of the two Hamiltonians H_0 and H can be very different, even though they are related by a canonical transformation: this is because of the explicit time-dependence of $\mathscr{F}$.
3. It is trivial to see that time-dependent perturbation theory works satisfactorily in two situations where conventional perturbation theory is singular:

a. Let H_1 be a function of I alone. Conventional theory is singular in this case, as the only non-zero Fourier component corresponds to $n = 0$. Our theory allows for a regular solution, $F_1 = H_1 t$, which yields,

$$J = I; \quad U = T + \varepsilon H_1; \quad u = t; \quad \phi = \theta - \varepsilon \frac{\partial H}{\partial I}; \tag{35}$$

where (I, T, t, θ) denote the original canonical coordinates and (J, U, u, ϕ) the transforms of these coordinates.

b. Let H_1 be such that H_1 explicitly depends on θ and $\{H_0, H_1\} = 0$. (Note: this can happen only if the unperturbed frequencies ω are constant.) Once again, we have the solution $F = H_1 t$ and it is easy to check the correctness of the transformed invariants.

4. As the examples described above clearly show, the canonical transformation defined by $\mathscr{F}$ changes not only the (usual) canonical coordinates (I, θ) but also T. It is the motions defined by H'_0 and H' in the extended phase space defined by (I, θ, T, t) which are being mapped into one another. In principle, we could try to reduce the transformation to one on the usual phase space by eliminating t (using, e.g., the invariant corresponding to T); it is fairly straightforward to see that this results in recovering the singularities of canonical perturbation theory.

Numerical studies of the application of this theory will be studied in companion papers.

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