PRAMANA
— journal of
physics

© Printed in India

Vol. 44, No. 6,
June 1995
pp. 511–518

Cosmic strings in Bianchi III spacetime: Integrable cases

S D MAHARAJ[1,2], P G L LEACH[1,2,3] and K S GOVINDER[1,2,3]
[1]Department of Mathematics and Applied Mathematics, University of Natal, Private Bag X10, Dalbridge 4014, South Africa
[2]Visiting: Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India
[3]Current address: Department of Mathematics, University of the Aegean, Karlovassi 83 200, Greece

MS received 28 February 1995

Abstract. We investigate the integrability of cosmic strings in Bianchi III spacetime using a symmetry analysis. The behaviour of the model is reduced to the solution of a single second order nonlinear differential equation. We show that this equation has a rich structure and admits an infinite family of solutions. Our class of solutions extends special cases previously obtained by Tikekar and Patel [*Gen. Relativ. Gravit.* **24**, 397 (1992)].

Keywords. Cosmology; strings; integrable.

PACS Nos 04·20; 98·90

1. Introduction

The study of cosmic strings is of considerable importance because it may generate density perturbations thereby leading to the formation of galaxies. (For details the reader is referred to the paper by Zel'dovich [1].) It has been noted that the presence of strings in the early universe does not contradict present day observations of the universe [2]. There is no direct evidence of strings in the present day universe; our models must therefore evolve from an era dominated by strings to an era dominated by particles which may possess remnants of strings. Cosmic strings have stress energy and couple in a simple way to the gravitational field. Most analyses are concerned with the gravitational effects which arise from the presence of strings.

Vilenkin [3], Gott [4] and Garfinkle [5] made pioneering analyses of the features of strings. The general relativistic formalism of cosmic strings is due to Letelier [6] and Stachel [7]. Subsequently, a number of general relativistic exact solutions were obtained in models of different Bianchi types. Krori *et al* [8] considered models with Bianchi types II, VI_0, VIII and IX. Bianchi type I string cosmological models were studied by Banerjee *et al* [9]. More recently a number of exact solutions, in the presence of a magnetic field and also with vanishing electromagnetic tensor, in Bianchi III spacetimes were obtained by Tikekar and Patel [10].

In this paper we consider other classes of string solutions in Bianchi III spacetime, thereby extending the Tikekar and Patel [10] solution. Essentially the solution of the field equations reduces to solving a single second order nonlinear ordinary differential equation. We show that this equation has a rich structure and admits an infinite number of solutions, some of which may lead to physically viable models. In § 2 we briefly outline the general field equations governing the behaviour of strings. The second order equation that has to be integrated is generated in § 3. The Lie symmetries of this equation are obtained and we demonstrate that it is transformable to the Emden–Fowler equation. In § 4 the field variables are presented in a general form. A comprehensive analysis is performed in § 5 of the fundamental integral constraint that underpins this analysis. Some concluding remarks are made in § 6. Throughout we follow the notations and conventions of Tikekar and Patel [10].

2. The field equations

We consider the general Bianchi III type spacetime with the line element

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)e^{-2ax}dy^2 - C^2(t)dz^2, \tag{2.1}$$

where a is a constant. The energy-momentum tensor is given by

$$T_{ik} = \rho u_i u_k - \lambda w_i w_k + \frac{1}{4\pi}\left[-g^{lm}F_{il}F_{km} + \frac{1}{4}g_{ik}F_{lm}F^{lm}\right] \tag{2.2}$$

for a cloud of string dust with a magnetic field along the z-direction. In (2.2) ρ, the proper energy density for a cloud of strings, and λ, the string tension density, are related by

$$\rho = \rho_p + \lambda, \tag{2.3}$$

where ρ_p is the particle density of the configuration. The quantity u^i is the four-velocity of the particles and w^i represents the direction of the string; these satisfy

$$u^i u_i = -w^i w_i = 1, \quad u^i w_i = 0.$$

We consider a comoving coordinate system so that

$$u^i = (0, 0, 0, 1), \quad w^i = (0, 0, 1/C, 0).$$

Tikekar and Patel [10] were concerned with Bianchi III symmetry and so made the assumption $A = B$.

As the magnetic field is along the z-direction, Maxwell's equations imply that

$$F_{12} = Ke^{-ax}, \tag{2.4}$$

where K is a constant, is the only nonzero component of the electromagnetic field tensor F_{ik}. We are now in a position to generate the Einstein field equations

$$R_{ik} - \tfrac{1}{2}Rg_{ik} = -8\pi T_{ik} \tag{2.5}$$

explicitly with the assistance of (2.2) and (2.4) for the line element (2.1). The field equations (2.5) reduce to the system

$$8\pi\rho = 2\frac{\dot{A}\dot{C}}{AC} + \frac{\dot{A}^2}{A^2} - \frac{a^2}{A^2} - \frac{K^2}{A^4} \tag{2.6}$$

$$8\pi\lambda = 2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} - \frac{a^2}{A^2} - \frac{K^2}{A^4} \tag{2.7}$$

$$0 = \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{K^2}{A^4}. \tag{2.8}$$

The particle density is given by

$$8\pi\rho_{\mathrm{p}} = -2\frac{\ddot{A}}{A} + 2\frac{\dot{A}\dot{C}}{AC}. \tag{2.9}$$

Note that (2.3) and (2.9) are consistent. We generate an infinite family of solutions to (2.6–2.8) in subsequent sections.

3. The model equation

The three nonvanishing field equations (2.6–2.8) contain the four dependent field variables ρ, λ, A and C. To enable explicit solutions determination a constraint must be imposed. Tikekar and Patel [10] take

$$C = A^n, \tag{3.1}$$

where n is a real constant, so that (2.8) becomes

$$(n+1)\frac{\ddot{A}}{A} + n^2\frac{\dot{A}^2}{A} + \frac{K^2}{A^4} = 0 \tag{3.2}$$

for which they were able to provide some solutions. It is a trivial matter to show that (3.1) is the most general constraint on the relationship between A and C for (2.8) to reduce to (3.2). Tikekar and Patel obtained exact solutions for the cases $n = 0$, $n = (3 \pm \sqrt{21})/2$, $n = (3 \pm \sqrt{33})/4$ and $n = 1 \pm \sqrt{3}$.

By inspection (3.2) is seen to have the two Lie point symmetries

$$G_1 = \frac{\partial}{\partial t} \tag{3.3}$$

$$G_2 = 2t\frac{\partial}{\partial t} + A\frac{\partial}{\partial A} \tag{3.4}$$

and it can be quickly verified (for example by the use of Program LIE [11]) that these are the only two Lie point symmetries except for the particular values $n = 0$, $(3 \pm \sqrt{21})/2$ and -1 for which the numbers of point symmetries are three, eight and infinity respectively. The existence of the additional symmetries is not necessary for the reduction of (3.2) to quadratures via the normal subgroup, G_1. (Since $[G_1, G_2] = G_1$, where $[G_1, G_2] = G_1G_2 - G_2G_1$ is the standard Lie Bracket, G_1 is the normal subgroup and this is the symmetry which should be used to reduce the order of (3.2) [12, p 148]. Reduction via G_2 leads to an Abel's equation of the second kind from which little joy can be expected.)

A more satisfactory solution is obtained if (3.2) is transformed to the autonomous Emden–Fowler equation (13, 15] of index ν, viz.

$$Y'' = \tfrac{1}{2}\varepsilon(\nu + 1)Y^{\nu}, \tag{3.5}$$

where $\varepsilon = \pm 1$ and the constant $(\nu + 1)/2$ is introduced with the advantage of hindsight. The algebra of G_1 and G_2 is A_2 in the Mubarakzyanov classification scheme [16]. Since it is solvable and $G_1 \not\subset G_2$, the algebra is that of Lie's Type IV [17, p 424]. However, we do not use the standard representation of a second order differential equation invariant under a type IV algebra since the form (3.5) is more suitable for the present discussion. The necessary tranformation is

$$t=\alpha X, \quad A=Y^{(n+1)/(n^2+n+1)}, \tag{3.6}$$

where

$$\alpha^2=-\frac{\varepsilon(n+1)^2(n^2-n-1)}{2K^2(n^2+n+1)^2}, \quad \nu=\frac{n^2-3n-3}{n^2+n+1}. \tag{3.7}$$

The case $\nu=-1$ is special and (3.7a) must be replaced by

$$\alpha^2=-\frac{\varepsilon(n+1)^2}{K^2(n^2+n+1)}. \tag{3.8}$$

The equation is

$$Y''=\varepsilon Y^{-1}. \tag{3.9}$$

In (3.5), $\varepsilon=-1$ for $n^2-n-1\geqslant 0$ and $\varepsilon=+1$ for $n^2-n-1\leqslant 0$. The latter applies to the interval $n\in((1-\sqrt{5})/2, (1+\sqrt{5})/2)$. The asymptotic value of ν is 1. It takes a minimum value of -3 at $n=0$. This corresponds to a special case of the Ermakov–Pinney equation [18, 19]. The maximum value of 7/3 is attained when $n=-2$. The parameter ν can take on all values in $[-3, 7/3]$. We shell see that in this interval there are many values of ν, hence n, for which the reduction of (3.2) to quadrature can be taken the further step of evaluation of the quadrature. The particular values of n given by Tikekar and Patel [10] for which explicit integration is possible correspond to $\nu=-3, 0, -3/5$ and $-1/3$. The case $n=-1$ is trivial.

The reduction of (3.5) to quadrature is elementary. The first integral is

$$I=\begin{cases}\frac{1}{2}Y'^2-\frac{1}{2}\varepsilon Y^{\nu+1}\\ \frac{1}{2}Y'^2-\log Y, \quad \nu=-1.\end{cases} \tag{3.10}$$

The solution is found by inversion of the result obtained after performance of the integration

$$X-X_0=\begin{cases}\int[2I+\varepsilon Y^{\nu+1}]^{-1/2}\mathrm{d}Y\\ \int[2I-2\log Y]^{-1/2}\mathrm{d}Y, \quad \nu=-1.\end{cases} \tag{3.11}$$

4. General results

Whatever the outcome of the quadratures in (3.11) we can write down some general results. (We omit the special case $\nu=-1$ to avoid what is virtually repetition.) Thus the metric (2.1) is

$$\begin{aligned}\mathrm{d}s^2=\mathrm{d}t^2&-\left[Y\left(\frac{t}{\alpha}\right)\right]^{2(n+1)/(n^2+n+1)}\mathrm{d}x^2\\ &-\mathrm{e}^{-2\alpha x}\left[Y\left(\frac{t}{\alpha}\right)\right]^{2(n+1)/(n^2+n+1)}\mathrm{d}y^2\\ &-\left[Y\left(\frac{t}{\alpha}\right)\right]^{2n(n+1)/(n^2+n+1)}\mathrm{d}z^2,\end{aligned} \tag{4.1}$$

the energy density (2.6) is

$$8\pi\rho = -\frac{K^2[2(2n+1)+(n^2-n-1)]}{(n^2-n-1)\,Y^{4(n+1)/(n^2+n+1)}} - \frac{4K^2I(2n+1)}{\varepsilon(n^2-n-1)\,Y^2} - \frac{a^2}{Y^{2(n+1)/(n^2+n+1)}}, \tag{4.2}$$

the string tension (2.7) is

$$8\pi\lambda = \frac{4IK^2(2n^2-n-1)}{\varepsilon(n+1)(n^2-n-1)\,Y^2} - \frac{K^2(n^2-n-3)}{(n^2-n-1)\,Y^{4(n+1)/(n^2+n+1)}} - \frac{a^2}{Y^{2(n+1)/(n^2+n+1)}} \tag{4.3}$$

and the particle density (2.9) is

$$8\pi\rho_{\mathrm{p}} = -\frac{8IK^2n(2n+1)}{\varepsilon(n+1)(n^2-n-1)\,Y^2} - \frac{4K^2(n+1)}{(n^2-n-1)\,Y^{4(n+1)/(n^2+n+1)}}. \tag{4.4}$$

The expansion scalar is

$$\theta = 2\frac{\dot{A}}{A} + \frac{\dot{C}}{C} = \frac{2K(n+2)}{[-2\varepsilon(n^2-n-1)]^{1/2}}\left[\frac{2I}{Y^2} - \frac{\varepsilon}{Y^{4(n+1)/(n^2+n+1)}}\right]^{1/2} \tag{4.5}$$

and the shear scalar is

$$\sigma^2 = \frac{2}{3}\left[\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right]^2 = -\frac{4K^2(n-1)^2}{3\varepsilon(n^2-n-1)}\left[\frac{2I}{Y^2} + \frac{\varepsilon}{Y^{4(n+1)/(n^2+n+1)}}\right]. \tag{4.6}$$

The expressions (4.1–4.6) give the physical parameters of interest once Y is known. We note the appearance of the constant of integration, I. In (3.10) we can take $I=0$ as a special case when $\varepsilon=+1$ which occurs for $n^2-n-1\leqslant 0$, i.e. $n\in((1-\sqrt{5})/2, (1+\sqrt{5})/2)$. This corresponds to $\nu\in[-3,-1]$. (Recall that ν takes its minimum at $n=0$.) Thus (3.10) reduces to

$$Y'^2 = Y^{\nu+1}. \tag{4.7}$$

We assume that $A(t)$ and $Y(X)$ are non-negative. Then (4.7) is easily integrated to

$$Y = \left[\frac{1-\nu}{2}(X-X_0)\right]^{2/(1-\nu)} \tag{4.8}$$

provided $\nu \neq -2$. (For the general discussion we have already excluded $\nu = -1$.) This gives a range of solutions for the field variables for n in the interval specified.

5. Solutions of the governing equation

The solution of the field equations (2.6–2.8) and the evaluation of the field variables etc. (4.1–4.6) have been reduced under the assumption (3.1) to the evaluation of the integral (3.11), viz.

$$X - X_0 = \begin{cases} \int [2I + \varepsilon Y^{\nu+1}]^{-1/2} \mathrm{d}Y \\ \int [2I - 2\log Y]^{-1/2} \mathrm{d}Y, \quad \nu = -1. \end{cases} \tag{5.1}$$

It is well-known that (5.1b) cannot be evaluated in closed form under any circumstances since it is a variant of the exponential-integral function [20, p 93]. However, (5.1a) is known to be evaluable as a standard integral for $\nu = -3, -2, 0, 1, 2, 3$ (the corresponding values for n are $0, (1 \pm \sqrt{13})/6, (3 \pm \sqrt{21})/2, -1, (-5 \pm \sqrt{5})/2, (-3 \pm i\sqrt{3})/2$.

The two cases given by Tikekar and Patel [10] which are not included in this list correspond to $\nu = -3/5$ $(n = (3 \pm \sqrt{33})/4)$ and $\nu = -1/3$ $(n = 1 \pm \sqrt{3})$. These values give a hint to one sequence for which (5.1a) can be evaluated in closed form. Let

$$\nu = -\frac{m}{m+2}, \quad m \in \mathscr{Z}. \tag{5.2}$$

Then (5.1a) is

$$X - X_0 = \int [2I + \varepsilon Y^{2/(m+2)}]^{-1/2} \mathrm{d}Y. \tag{5.3}$$

For $m + 2 \geqslant 0$ the first nontrivial value of m is $m = 1$. For $m \geqslant 1$, $\nu \in (-1, -1/3)$. Over this interval $\varepsilon = -1$. The appropriate substitution is

$$Y = [(2I)^{1/2} \sin u]^{m+2} \tag{5.4}$$

so that (5.3) becomes

$$(X - X_0) = (m+2)(2I)^{(m+4)/2} \int^{u(Y)} \sin^{m+1} u \mathrm{d}u \tag{5.5}$$

which can be evaluated in closed form for all integral $m \geqslant 1$. Inversion is not generally possible apart from locally. The one exception is $m = 2 (n = (5 \pm \sqrt{85})/6)$. However, (5.4) with (5.5) does define a parametric solution. It is a simple matter to express (4.1–4.6) in terms of the parameter u through (5.4)

For $m + 2 \leqslant 0$ we replace m by $-p-2$, $p \in \mathscr{Z}^+$. Then (5.3) becomes

$$X - X_0 = \int [2I + \varepsilon Y^{-2/p}]^{-1/2} \mathrm{d}Y \tag{5.6}$$

and

$$\nu = -\frac{p+2}{p} \tag{5.7}$$

so that $\varepsilon = +1$ for all admissible values of p. We rewrite (5.6) as

$$X - X_0 = \int Y^{1/p} [2I\, Y^{2/p} + 1]^{-1/2} \mathrm{d}Y \tag{5.8}$$

and the integral is evaluated in closed form by the substitution

$$Y = [(2I)^{-1/2} \sinh u]^p. \tag{5.9}$$

Finally we note that there is another set of values of ν for which (3.5) is integrable. If

$$\nu = \frac{p+2}{p}, \quad p \in \mathscr{Z}^+, \tag{5.10}$$

(3.5) possesses the Painlevé Property [21] and is integrable in the sense of Painlevé [22]. Unfortunately the evaluation of the quadrature is by no means obvious. As a matter of curiosity we note that when ν is given by (5.7), the negative of (5.10), (3.5) does not possess the Painlevé Property in its present form, but the quadrature of (3.11) is straightforward.

6. Conclusion

In this paper we have shown, using symmetry arguments, how the equation governing the behaviour of cosmic strings in Bianchi III spacetime may be reduced to the quadrature (3.11). A detailed analysis of (3.11) was performed. Those cases for which (3.11) is evaluable as a simple integral, from standard handbooks of integrals, are given; particular cases reported previously in [10] are identified. In addition, we present a particular sequence for which the integral may be evaluated in closed form. In general for this sequence the solution can only be put into parametric form and inversion is only possible locally. Few closed form solutions to the Einstein equations for cosmic strings have been reported in the literature. Our analysis is an attempt to obtain more exact solutions so that our understanding of these objects may be improved. It is hoped that some of the solutions presented here will provide the basis for a detailed physical analysis of cosmic strings in the gravitational context.

Acknowledgements

Authors thank the Foundation for Research Development of South Africa and the University of Natal for their continued support. Authors thank the Foundation for Research Development of South Africa and the University of Natal for their continued support.

References

[1] Ya B Zel'dovich, *Mon. Not. R. Astron. Soc.* **192**, 663 (1980)
[2] T W S Kibble, *J. Phys.* **A9**, 1387 (1976)
[3] A Vilenkin, *Phys. Rev.* **D24**, 2982 (1981)
[4] J R Gott, *Astrophys. J.* **288**, 422 (1985)
[5] D Garfinkle, *Phys. Rev.* **D32**, 1323 (1985)
[6] P S Letelier, *Phys. Rev.* **D20**, 1294 (1979)
[7] J Stachel, *Phys. Rev.* **D21**, 2171 (1980)
[8] K D Krori, T Chaudhuri, C R Mahanta and A Mazumdar, *Gen. Relativ. Gravit.* **22**, 123 (1990)
[9] A Banerjee, A K Sanyal and S Chakrabarty, *Pramana – J. Phys.* **34**, 1 (1990)
[10] Ramesh Tikekar and L K Patel, *Gen. Relativ. Gravit.* **24**, 397 (1992)
[11] A K Head, *Comp. Phys. Commun.* **77**, 241 (1993)
[12] Peter J Olver, *Applications of Lie groups to differential equations*, second edition (New York, Springer-Verlag, 1993)
[13] I J Homer Lane, *Am. J. Sci. Arts* **4**, 57 (1869–1870)

[14] R Emden, *Gaskugeln, Anwendungen der mechanischen Warmen-theorie auf Kosmologie und meteorologische Probleme*, Leipzig, Teubner (1907)
[15] R H Fowler, *Q. J. Math.* **45**, 289 (1914);
R H Fowler, *Mon, Not. R. Astron. Soc.* **91**, 63 (1930)
[16] J Patera and P Winternitz, *J. Math. Phys.* **18**, 1449 (1977)
[17] S Lie, *Differentialgleichungen*, (New York, Chelsea, 1967)
[18] V Ermakov, *Univ. Izv. Kiev Ser. III* **9**, 1 (1880)
[19] Edmund Pinney, *Proc. Am. Math. Soc.* **1**, 681 (1950)
[20] I S Gradshteyn and I M Ryzhik, *Table of integrals, series and products.* fourth edition, edited by Allan Jeffrey (Academic Press, San Diego, 1980)
[21] K S Govinder and P G L Leach, *Integrability analysis of the Emden–Fowler equation*, Preprint: Department of Mathematics, University of the Aegean (1994)
[22] P Painlevé, *Acta Math.* **25**, 1 (1902)