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Supersymmetry in complex space-time

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Abstract. An interconnection between superluminal transformation and supersymmetric transformations has been investigated in complex C^3-space and the evolution of bosonic and fermionic subspaces in such space has been undertaken. Introducing the suitable anticommuting operators to induce grading in Poincare group in C^3-space in terms of components of complex angular momentum operator, the supersymmetric algebra connecting bradyonic and tachyonic bosons and fermions has been constructed and it has been demonstrated that the difference between scales of bosonic and fermionic subspaces in C^3-space increases quickly in spite of their closeness initially.

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1. Introduction

Basic algebraic structure underlying the notion of supersymmetry is a graded Lie algebra which involves both commutation and anticommutation relations and plays a unique role in particle physics providing a fusion between space-time and internal symmetries overcoming no-go theorem [1]. In the history of diversified ways of the study of various space-time symmetries and their fusion with internal symmetries, one of the convenient methods has been to place emphasis on homogeneous Lorentz group (HLG) which leads to the study of the irreducible representations of Poincare group used as Lie algebra in super-symmetric theories. Keeping in view the importance of reconstruction and investigation of the representations of SL (2, C) group, universal covering group of which is HLG, a compact operator formulation has been developed [2] in explicit continuable form to reformulate the Gel'fand–Naimark theory [3]. Though the space-time representations of Poincare group have made their appearance in several investigations of mathematical and physical nature [4, 5], the space-like objects have been shown [6, 9] to suffer with the problem of proper representation and localization in usual four-dimensional space $R^4(\mathbf{r}, t)$ and it has been demonstrated [10] that the relativistic extension to superluminal phenomena and the building of unified theory of bradyons [11] and tachyons [12] are possible only in a pseudo-Euclidean higher dimensional space-time $D^8 = R^4(\mathbf{r}, t) \times T^4(\mathbf{r}, t)$, with equal number of spatial and temporal constituents. Such higher dimensional symmetrical space has been complexified by various authors [13–15] and it has been demonstrated that the transformation in the complex space C^3, consisting of real, spatial and temporal three vectors are related to the group SU(3) of unitary intrinsic symmetries of elementary particles and law of trichromatism.

Justification for the need of complex space-time was put forward by various authors in special relativity [16], general relativity [17] and Twistor theory [17, 18]. In our recent work [19] we have analysed subluminal and superluminal Lorentz transformations in such a complex C^3-space, constructed [20] the complex angular momentum operators for bradyons and tachyons and derived [21] the realizations of HLG for non-zero real mass, zero mass and imaginary mass (i.e. bradyons, photons and tachyons) systems. An attractive interconnection of such an extended manifold with superluminal theories makes the perspective to invoke a variety of ideas of many dimensional schemes as the basis for unified theory of all physical interactions in space-time of non-trivial topology. Separation of extra time-like dimensions from space-like ones leads to the connection of superluminal transformations and supersymmetric transformations [22, 23], where, the former tranforms bradyonic objects into tachyonic ones and vice versa, and the latter transforms boson states into fermionic states.

In the present paper, the interconnection between superluminal transformations and supersymmetric transformations has been investigated in complex C^3-space and the study of evolution of bosonic and fermionic subspaces has been undertaken. Introducing the suitable anticommuting operators to induce grading in Poincare group in C^3-space in terms of components of complex angular momentum operator, the supersymmetric algebra connecting bradyonic and tachyonic bosons and fermions has been constructed in terms of spinorial charges giving rise to close system of commutation and anticommutation relations. Introducing eight dimensional theory with supersymmetric coordinates in C^3-space, the study of evolution of bosonic and fermionic subspaces and their interaction has been undertaken and it has been demonstrated that the difference between scales of bosonic and fermionic subspaces in C^3-space increases quickly in spite of their closeness initially.

2. Generators of Lorentz group in complex space

In complex three-space C^3, consisting of symmetrical, spatial and temporal components, an event A is specified by three complex coordinates,

$$\{A\} = (z^1, z^2, z^3) \tag{2.1}$$

where

$$z^j = x^j + it^j, \quad (j = 1, 2, 3) \tag{2.2}$$

Using natural units $c = \hbar = 1$, and the convention that $x^j = -x_j, t^j = t_j$, the interval between two events in this space may be written as,

$$ds^2 = \mathrm{Re}(dz^j \cdot dz_j^*) \tag{2.3}$$

where Re denotes the real part and * denotes the complex-conjugate. The generalized linear momentum corresponding to translation in C^3-space may be defined as

$$\mathbf{P} = \mathbf{p} + i\mathbf{E} \tag{2.4}$$

where the usual vector momentum $\mathbf{p}$, denotes translations along spatial coordinates and the vector energy $\mathbf{E}$, denoting the translations along the temporal coordinates, is directed tangentially to the time trajectory. Generalized angular momentum operator, corresponding to relations in C^3-space, may be constructed in the following manner:

$$\hat{\mathscr{Z}} = \hat{J} + i\hat{K} \tag{2.5}$$

where $\hat{J}$ generates pure rotations (spatial as well as temporal) and $\hat{K}$ generates space-time rotations (boosts). The Hamiltonian of a free particle in C^3-space may be constructed as

$$\hat{H} = \frac{\hat{P}^2}{2|M_c|} \tag{2.6}$$

where the proper mass M_c of the particle in C^3-space is also a complex quantity

$$M_c = m + i\mu. \tag{2.7}$$

Assuming the orthogonality between vectors **p** and **E** in (2.4), we may write the modulus of this Hamiltonian as

$$|H| = (p^2 - E^2)/2|M_c| \tag{2.8}$$

which reduces to following forms, respectively, in the four-spaces $R^4(\mathbf{r}, t)$ with $t = (t_1^2 + t_2^2 + t_3^2)^{1/2}$ and, $T^4(\mathbf{t}, r)$ with $r = (x_1^2 + x_2^2 + x_3^2)$;

$$H_B = p^2/2m$$

and

$$H_T = -E^2/2\mu \tag{2.9}$$

where B and T denotes bradyons and tachyons with the natural spaces for their physical specifications as R^4 and T^4 respectively.

Operators $\hat{P}$, $\hat{\mathcal{Z}}$ and $\hat{H}$ given by (2.4), (2.5) and (2.6) constitute generalized Poincare group in C^3-space and satisfies the following commutation rule,

$$[\hat{H}, P] = 0; \quad [\hat{H}, \hat{Z}_j] = 0, \quad [H, \mathcal{Z}^2] = 0$$
$$[\hat{\mathcal{Z}}_j, \hat{\mathcal{Z}}_k] = i\varepsilon_{jkl} Z_l. \tag{2.10}$$

3. Global supersymmetry in C^3-space

Basic algebraic structure underlying the notation of supersymmetry is graded Lie algebra which is a graded extension of Poincare group. This grading makes a distinction between even and odd elements where, even elements, belonging to Lie algebra, obey commutation relations and the odd elements which are responsible for the grading, obey anticommutation relations among themselves and commutation relations with the even elements. The representation of Lie algebra in terms of odd elements is known as grading representation. For the Poincare group specified by (2.10) in C^3-space, let us introduce grading by the anticommuting operators

$$Q'_\alpha = Q_\alpha + i q_\alpha \tag{3.1}$$

such that

$$\{Q_\alpha, Q_\beta\} = -2(\gamma^j k^{-1})_{\alpha\beta} P_j$$
$$\{q_\alpha, q_\beta\} = -2i(\gamma^j k^{-1})_{\alpha\beta} E_j \tag{3.2}$$
$$\{q_\alpha, Q_\beta\} = 0$$

where k is the charge conjugation matrix. Substituting the usual relation

$$\bar{Q} = \tilde{Q}\gamma^0$$

equations (3.2) may be written as

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = -2\gamma^j_{\alpha\beta} p_j; \\ &\{q_\alpha, \bar{q}_\beta\} = 2i\gamma^j_{\alpha\beta} E_j \\ &\{q_\alpha, \bar{Q}_\beta\} = 0; \quad \{\bar{q}_\alpha, Q_\beta\} = 0. \end{aligned} \tag{3.3}$$

Then we may construct the following graded algebra ($N = 1$ supersymmetry) as the graded extension of generalized Poincare group defined by (2.10);

$$\begin{aligned} &[Q'_\alpha, \mathcal{Z}^l] = i\sigma^{jk}_{\alpha\beta} Q'_\beta \\ &[Q'_\alpha, \bar{Q}'_\beta] = -2\gamma^j_{\alpha\beta} P_j \\ &[P_j, Q'_\alpha] = 0 \end{aligned} \tag{3.4}$$

where $\sigma^{jk} = 1/4\,[\gamma^j, \gamma^k]$, and $j,k,l = 1,2,3$ with $j,k \neq l$. These equations lead to

$$\begin{aligned} &[p_j, Q_\alpha] = 0; \quad [P_j, q_\alpha] = 0 \\ &[E_j, q_\alpha] = 0; \quad [E_j, Q_\alpha] = 0 \\ &[q_\alpha, \mathcal{Z}^l] = i\sigma^{jk}_{\alpha\beta} q_\beta - i\sigma^{01}_{\alpha\beta} Q_\beta \\ &[Q_\alpha, \mathcal{Z}^l] = i\sigma^{jk}_{\alpha} Q_\beta - \sigma^{01}_{\alpha\beta} Q_\beta \end{aligned} \tag{3.5}$$

which may be further reduced to the graded algebras in R^4 and T^4 spaces separately. The graded algebra (3.4) may therefore be treated as supersymmetry which connects bradyonic and tachyonic, bosonic and fermionic fields. This supersymmetry of bosons and fermions, with bradyons and tachyons on equal footing, is generated by charges transforming like spinors under generalized Poincare group defined by (2.10). These spinorial charges give rise to a closed system of commutation and anticommutation relations (3.5) and (3.3) which may be called as pseudo-Lie algebra. Any linear representation of supersymmetric generators of (3.4) contains both fermions (bradyonic as well as tachyonic) and bosons (bradyonic as well as tachyonic) which have equal masses (magnitudes) if the symmetry is unbroken.

4. Evolution of supersymmetric subspaces in C^3-space

Many different dimensional schemes are thought to be the basis for building the unified theory of all physical interactions (including the tachyonic ones). In these extended manifolds the signs of metrical coefficients are interchanged on passing horizon [24] and there are many dimensional regions of such representation in which time-like and space-like coordinates are taken into account in unified symmetrical way. The introduction of additional (extra) dimensions with the unified consideration of bradyonic and tachyonic subspaces, R^4 and T^4 respectively in C^3-space, allows to overcome some problems of trivial construction with time-like coordinates in extra space-time manifolds. Let us consider the eight dimensional theory with supersymmetric coordinates in C^3-space to study the evolution of bosonic and fermionic subspaces and their interactions on the basis of general relativity (with supersymmetry between four-dimensional subspaces R^4 and T^4). The bosonic and fermionic dimensions form the global eight dimensional space-time manifold,

$$(M,N)=(t,z^1,z^2,z^3,\bar{t},\bar{z}^1,\bar{z}^2,\bar{z}^3) \tag{4.1}$$

where

$$t=(t_x^2+t_y^2+t_z^2)^{1/2},$$

and

$$z^j=x^j+it^j,\quad (j=1,2,3)$$

and the fermionic coordinates $(\bar{t},\bar{z}_i)$ are transformed from the bosonic one (t,z_i) by means of the following transformation

$$\bar{z}^j\bar{z}^j=z^\alpha h_{\alpha\beta}z^\beta \tag{4.2}$$

with

$$h=\begin{bmatrix}\sigma_2 & & \\ & \ddots & \\ & & \sigma_2\end{bmatrix} \tag{4.3}$$

σ_2 being the Pauli matrix. Matter dynamics may be investigated in this model by using the uniform and isotropic spherical symmetrical metric for subspaces which are reduced to the Friedmann–Robertson Walker metric in the four-dimensional case. It admits either matter or field in the right part of Einstein–Maxwell equations. Thus we have,

$$G_{MN}=\begin{bmatrix}-1 & & & \\ & g_{ij}R^2 & & \\ & & +1 & \\ & & & g_{mn}\bar{R}^2\end{bmatrix} \tag{4.4}$$

where R and $\bar{R}$ are maximally symmetric subspaces.

The line element may be written as follows,

$$\begin{aligned}\mathrm{d}s^2= & -\mathrm{d}t^2+\mathrm{d}\bar{t}^2+b^2\mathrm{d}z^\alpha\mathrm{d}z^{*\alpha}(1+|z|^2/4)^{-2}\\ & -f^2\mathrm{d}\bar{z}^\alpha\mathrm{d}\bar{z}^{*\alpha}(1+|\bar{z}|^2/4)^{-2}\end{aligned} \tag{4.5}$$

while b and f denote the bosonic and fermionic scale factors. This line element may be written as follows in the supersymmetric representation

$$\mathrm{d}s^2=E^M G_{MN}E^{*N}$$

where Vielbein E^M is given by

$$E^1=\mathrm{d}t,\quad E^5=-\mathrm{d}t$$

$$(E^\alpha,E^\alpha)=\frac{(d_z^\alpha b,\mathrm{d}\bar{z}^{\alpha_f})}{[1+|z|^2/4]^2} \tag{4.6}$$

The non-zero components of G_{MN} are

$$G_{00} = 1, \quad G_{\alpha\alpha} = 1, \quad G_{55} = +1, \quad G_{\alpha\beta} = h_{\alpha\beta}$$

and we get the curvature invariant as

$$R/3 = 2(\ddot{b}/b - \ddot{f}/f) - 2\dot{b}^2/b^2 + 6\dot{b}\dot{f}/bf - 4\dot{f}^2/f^2$$
$$- 1/4[|z|^2 + |\bar{z}|^2 + 1][b^{-2} + f^{-2}] \tag{4.7}$$

Then action is giving by

$$S = \int d^8 y g^{1/2} R = \int dt\, L \tag{4.8}$$

where L is Lagrangian and d^8y is the element

$$dt\, d\bar{t}\, dz_1\, dz_2\, dz_3\, d\bar{z}_1\, d\bar{z}_2\, d\bar{z}_3.$$

Let us introduce the following parameters in terms of bosonic and fermionic scale factors,

$$Q = 1/2 \log(bf)$$

and

$$q = \log(b/f) \tag{4.8a}$$

then the action given by (4.8) yields the following equations of motion

$$3q + 27/4\dot{q}^2 + 3/2\dot{q}\dot{Q} + 3/2\exp(-2Q)\sinh q = 0$$
$$\ddot{q} + 3\dot{q}^2 + 2\exp(-2Q)\sinh q = 0 \tag{4.9}$$
$$\ddot{q} - 1/3\ddot{Q} + 3/2\dot{q}^2 + 1/3\exp(-2Q)[3\sinh q + \cosh q] = 0$$

These non-linear differential equations determine the evolution laws for bosonic and fermionic subspaces. An exact complete solution of these equations is very difficult by analytical methods. Let us consider the following approximate solutions without any loss of generality:

(a) Power law approximation

In a general case, (4.9) can also be written as,

$$\ddot{q} - 6\dot{q}^2 - 2\dot{q}\dot{Q} - 4q = 0 \tag{4.10}$$

and

$$\ddot{Q} + 9\dot{q}^2 + (6\sinh q - 2\cosh q)\exp(-2Q) = 0. \tag{4.11}$$

Let us assume the following power law dependence on time

$$q = Bt^{n_1} - Ft^{n_2} \tag{4.12}$$
$$Q = 1/2[Bt^{n_1} + Ft^{n_2}]$$

where B and F are positive constants and n_1 and n_2 are integer constants. Then (4.10) becomes

$$Bn_1(n_1-1)t^{n_1-2} - 7B^2n_1^2t^{2(n_1-1)} - 4Bt^{n_1} - Fn_2(n_2-1)t^{n_2-2} - 5F^2n_2^2t^{2(n_2-1)} + 4Ft^{n_2} + 12n_1n_2BFt^{n_1+n_2-2} = 0 \quad (4.13)$$

and (4.11) becomes

$$Bn_1(n_1-1)t^{n_1-2} + 9n_1^2B^2t^{2(n_1-1)} + n_2(n_2-1)Ft^{n_2-2} + 9n_2^2F^2t^{2(n_2-1)} - 18n_1n_2BFt^{n_1+n_2-2} - 2e^{-2Ft^{n_1}} - 4e^{-2Bt^{n_2}} = 0. \quad (4.14)$$

For $n_1 = n_2 = 1$, (4.13) gives

$$B = F$$

or

$$5F - 7B = 4t. \quad (4.15)$$

Substituting relations (4.8a) and (4.12) into (4.15), we get

$$b = f$$

or

$$f/b = b^{2/5}e^{4/5}t^2 \quad (4.16)$$

Conditions (4.15) and (4.16) show that either the bosonic and fermionic scales remain the same in time or the difference in scales of fermionic and bosonic subspaces increases with time (i.e. the ratio of f and b increases in time). This connection between these subspaces enables us to determine the properties of one subspace (fermionic) through another (bosonic) subspace. For this case (i.e. $n_1 = n_2 = 1$), (4.14) reduces to

$$9B^2 + 9F^2 - 18BF = 2[e^{-2Ft} + 2e^{-2Bt})$$

which gives

$$b/f = \exp[(\sqrt{2}t)/(3bf)(b^2 + 2f^2)^{1/2}] \quad (4.17)$$

which also gives the above mentioned connection between fermionic and bosonic subspaces in time. For the choice $n_1 = n_2 = 2$, (4.13) reduces to

$$(B - F)/2 = [B - F + 7B^2 + 5F^2 + 12BF]t^2 \quad (4.18)$$

which also gives a simple connection between B and F. It may also be written as,

$$(F - B)/(F + B) = 2(7B + 5F)[2 - 1/t^2]^{-1} \quad (4.19)$$

showing an increase in difference of bosonic and fermionic scales, when time increases beyond $t = 1/\sqrt{2}$. At sufficiently large time ($t \to \infty$), this difference maintains a constant value. Initially, for $t = 0$, $F = B$, i.e. bosonic and fermionic scales are identical.

For this choice, we have

$$B = 1/t^2 \log b$$

and

$$F = 1/t^2 \log f$$

and hence (4.18) may also be written as

$$\frac{\log b/f}{\log(bf)} = \frac{2}{1-2t^2}\log(b^7 f^5) \tag{4.20}$$

showing the same increase in the difference of scales of bosonic and fermionic subspaces with the increase in time in the interval $0 < t < 1/\sqrt{2}$.

(b) Approximation for small difference in b and f.

Let the difference of bosonic and fermionic scales be very small i.e.

$$q \gg \dot{q}.$$

Then (4.10) reduces to

$$\ddot{q} - 6\dot{q}^2 - 2\dot{q}\dot{Q} = 0 \tag{4.21}$$

Let us assume its solution in term of zeroth approximation for Q, i.e.

$$Q = \log c + D\log t \tag{4.22}$$

which yields

$$bf = c_1 t^{\gamma_1} \tag{4.23}$$

where $c_1 = c^2$ and $\gamma = 2D$, both are positive constants. Then (4.21) becomes

$$\ddot{q} - 6\dot{q}^2 - 2\dot{q}D/t = 0 \tag{4.24}$$

Let its solution be

$$q = \log c_2 + \log t^{\gamma} \tag{4.25}$$

which yields

$$b/f = c^2 t^{\gamma}.$$

Substituting this solution into (4.24) we get

$$\gamma = 0 \quad \text{or} \quad \gamma = -[(2D+1)/6] = -\gamma_2$$

where $\gamma_2 = (\gamma_1 + 1)/6$ is non-zero positive. For $\gamma = 0$ we may set $b = f$ which gives the initial stage. At a later time we have

$$b/f = c_2 t^{-\gamma_2}, \tag{4.26}$$

Combining solutions (4.23) and (4.26) we have

$$b = \sqrt{c_1 c_2}\, t^{(\gamma_1 - \gamma_2)/2}$$

and

$$f = \sqrt{\frac{c_1}{c_2}}\, t^{(\gamma_1 + \gamma_2)/2} \tag{4.27}$$

showing that the difference between bosonic and fermionic scales increases quickly in spite of their closeness initially. It shows the similarity of bosonic and fermionic evolution in time. Thus (4.27) shows that supersymmetry is broken rather weakly.

The unobservability of more than four-dimensions of space-time may be associated with the event horizon effect ($R^4 \leftrightarrow T^4$) taking into account the interconnection between boson–fermion symmetry and bradyon–tachyon transformations. Supersymmetric connections in higher dimensions ought to be perspective in order to invoke the deeply developed theory of supersymmetry in realistic physics. It may also be used for the explanation of absence of many supersymmetric partners for ordinary particles since various ions and *s*-quarks may possibly exist in additional dimensions and can manifest themselves only in the regions where different dimensions are mixed.

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