PRAMANA
— journal of
physics

Vol. 44, No. 6,
June 1995
pp. 495–500

Higher dimensional spherically symmetric inhomogeneous cosmological model with heat flow

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MS received 2 August 1994; revised 14 February 1995

Abstract. We study spherically symmetric inhomogeneous cosmological model with heat flow in higher dimensional space-time and present a class of solutions in which the velocity field is shear-free. Some of these solutions are analogous to the known solutions in 4-dimension while some are totally new.

Keywords. Cosmology; heat flux; higher dimension; spherically symmetric solutions.

PACS Nos 04·20; 95·30

1. Introduction

In the usual space-time dimension, Bergmann [1] initiated the study of cosmological solutions with an energy flux. Then several authors (see for example, Maiti [2], Modak [3], Banerjee *et al* [4], Deng [5]) have found other solutions for 4-dimensional space-time. The reason to study cosmological solutions with heat flow is that one can get a general picture of gravitational collapse of stars, considering dissipation in the form of radial heat flow in simple adiabatic fluid. Also thermodynamically, one can relate non-uniformity of temperature to heat flux. So a non-zero heat flux means a temperature gradient as may be expected to occur following the formation of a gravitational condensation. Further, heat conduction ensures entropy production on general thermodynamic grounds.

As far as our knowledge goes, there has been no study so far in higher dimensions considering heat flux. In this paper, we have found some cosmological solutions with heat flux in $(D+2)$-dimensional space-time. The idea behind the consideration of higher dimensional space-time is to consider the effect of heat flux at the very early Universe when the higher dimensional theory is very important. The paper is organized as follows: Section 2 deals with basic equations and the field equations are solved in different situations to obtain different sets of cosmological solutions. In § 3, we consider solutions corresponding to the isotropy of pressure. The paper ends with a short discussion and concluding remarks.

2. Basic field equations and their solutions

The metric for a higher dimensional, spherically symmetric non-static space-time can be taken in isotropic form as

$$dS^2 = e^{\gamma} dt^2 - e^{\omega}(dr^2 + r^2 d\Omega_D^2), \tag{1}$$

where $d\Omega_D^2$ is the metric on the D-sphere in polar co-ordinates with expression

$$d\Omega_D^2 = dx_1^2 + \sin^2 x_1 dx_2^2 + \sin^2 x_1 \cdot \sin^2 x_2 dx_3^2 + \cdots$$

and

$$\gamma = \gamma(r, t), \quad \omega = \omega(r, t).$$

So, here space-time is $(D+2)$-dimensional. The energy-momentum tensor for a fluid with heat flux is expressed in the standard form as

$$T_\mu^\gamma = (\rho + p) v_\mu v^\gamma - p \cdot \delta_\mu^\gamma + q^\gamma \cdot v_\mu + q_\mu \cdot v^\gamma, \tag{2}$$

where q^μ stands for heat flow vector, orthogonal to the velocity vector v^μ. In co-moving co-ordinate system $v^\mu = e^{-\gamma/2}\delta_0^\mu$, this ensures that the velocity field is shear-free. In the present spherically symmetric case the only non-vanishing component of heat flow vector is q^1, along the radial direction. The non-trivial components of the Einstein field equation

$$R_\mu^\gamma - \tfrac{1}{2} R\, \delta_\mu^\gamma = -8\pi\, T_\mu^\gamma,$$

for the metric (1) are [6]

$$e^{-\omega}\frac{D(D-1)}{2r^2} + e^{-\gamma}\frac{D(D+1)}{8}\dot{\omega}^2 - e^{-\omega}\left[D\left(\frac{\omega''}{2} + \frac{1}{r^2}\right) + \frac{D(D+1)}{2}\left(\frac{\omega'}{2} + \frac{1}{r}\right)^2 - \frac{D\omega'}{2}\left(\frac{\omega'}{2} + \frac{1}{r}\right)\right] = 8\pi\rho, \tag{3}$$

$$e^{-\gamma}\left[D\frac{\ddot{\omega}}{2} - D\frac{\dot{\omega}\dot{\gamma}}{4} + \frac{D(D+1)}{8}\dot{\omega}^2\right] + e^{-\omega}\frac{D(D-1)}{2r^2} - e^{-\omega}\left[D\frac{\gamma'}{2}\left(\frac{\omega'}{2} + \frac{1}{r}\right) + \frac{D(D-1)}{2}\left(\frac{\omega'}{2} + \frac{1}{r}\right)^2\right] = -8\pi p, \tag{4}$$

$$e^{-\gamma}\left[\frac{D\ddot{\omega}}{2} + \frac{D(D+1)}{8}\dot{\omega}^2 - \frac{D\dot{\omega}\dot{\gamma}}{4}\right] + e^{-\omega}\frac{(D-1)(D-2)}{2r^2} - e^{-\omega}\left[\frac{\gamma''}{2} + \frac{\gamma'^2}{4} + (D-1)\left(\frac{\omega''}{2} - \frac{1}{r^2}\right) + \frac{D(D-1)}{2}\left(\frac{\omega'}{2} + \frac{1}{r}\right)^2 + \frac{(n-1)}{2}(\gamma' - \omega') \times \left(\frac{\omega'}{2} + \frac{1}{r}\right) - \frac{\omega'\gamma'}{4}\right] = -8\pi p, \tag{5}$$

and

$$\frac{D}{2}(\dot{\omega}' - \dot{\omega}\gamma'/2)e^{-\omega} = 8\pi q^1 \cdot v_0. \tag{6}$$

Here, differentiation with respect to r and t are denoted by prime and overhead dot respectively. Let us define a pair of variables (u, v) as follows:

$$u = [e^\omega \cdot r^2]^\alpha \quad \text{and} \quad v = \ln r, \tag{7}$$

α is an arbitrary constant. Then (3) has the following compact form using the new variables from (7):

$$\frac{d^2u}{dv^2}=\frac{(D-1)^2}{4}u-\frac{(D-1)}{2D}u^{(D+3)/(D-1)}\left[8\pi\rho-\frac{D(D+1)}{8}e^{-\gamma}\dot{\omega}^2\right] \tag{8}$$

provided the arbitrary constant α is taken to be $(D-1)/4$. Now, in order to integrate (8) in close form, let us assume an expression for ρ in the form

$$8\pi\rho=\frac{D(D+1)}{8}e^{-\gamma}\dot{\omega}^2+\beta u^l, \tag{9}$$

where $\beta=\beta(t)$ and l are arbitrary. The elimination of ρ from (8) and (9) gives rise to the second order differential equation in u

$$\frac{d^2u}{dv^2}=au+bu^m \tag{10}$$

with

$$a=4\alpha^2,\quad m=1+l+1/\alpha,\quad b=-\beta\frac{(D-1)}{2D},\quad \alpha=\frac{D-1}{4}.$$

This can be solved at once to give the explicit form for ω as

$$e^{\omega}=R/[1+Kr^{\alpha}]^{4/\alpha}, \tag{11}$$

where R depends on α, l and K and $K=K(t)$ is arbitrary. To obtain the other metric coefficient we first eliminate p from (4) and (5) and then using (11) for ω the differential equation in γ is

$$\gamma''+\frac{\gamma'^2}{2}+\frac{\gamma'}{r}\frac{(3Kr\alpha-1)}{1+Kr^{\alpha}}+4\frac{K(D-1)(2-\alpha)r^{\alpha-2}}{(1+Kr^{\alpha})^2}=0. \tag{12}$$

In general, this cannot be integrated at once for arbitrary values of $\alpha(D)$ and K. So we shall study solutions for particular values of $\alpha(D)$ and K.

Case I: $\alpha=2$, K is arbitrary i.e. the space-time is 11-dimensional

In this case (12) can be solved easily to obtain

$$e^{\gamma}=\left[1+\frac{\gamma_0}{1+K\gamma^2}\right]^2, \tag{13}$$

γ_0 is an arbitrary function of t alone. This solution is analogous to the 4-dimensional solution of Modak [3]. The expressions for physical quantities namely ρ, p, q^1 and θ for the above solution are

$$8\pi\rho=\frac{180K}{R}+\frac{45}{4}\left[\frac{\dot{R}}{R}-\frac{2\gamma^2\dot{K}}{1+K\gamma^2}\right]^2\left(1+\frac{\gamma_0}{1+K\gamma^2}\right)^{-2},$$

$$8\pi p=\frac{144K}{R}-\frac{2\gamma_0K}{R}\left(\frac{1-K\gamma^2}{1+K\gamma^2}\right)\left(1+\frac{\gamma_0}{1+K\gamma^2}\right)^{-1}$$
$$-\frac{45}{4}\left(\frac{\dot{R}}{R}-\frac{2\gamma^2\dot{K}}{1+K\gamma^2}\right)^2\left(1+\frac{\gamma_0}{1+K\gamma^2}\right)^{-2}$$

$$-\frac{27}{8}\left[\frac{\dot{R}}{R}-\frac{2\gamma^2\dot{K}}{1+K\gamma^2}\right]^{-1}\frac{\partial}{\partial t}\left\{\left[\frac{\dot{R}}{R}-\frac{2\dot{K}\gamma^2}{1+K\gamma^2}\right]^2\left(1+\frac{\gamma_0}{1+K\gamma^2}\right)^{-2}\right\}, \tag{14}$$

$$8\pi q' = -\frac{18\dot{K}\gamma}{R}\left(1+\frac{\gamma_0}{1+K\gamma^2}\right)^{-1}+\frac{9\gamma_0 K\gamma}{R}\left(\frac{\dot{R}}{R}-\frac{2\dot{K}\gamma^2}{1+K\gamma^2}\right)\left(1+\frac{\gamma_0}{1+K\gamma^2}\right)^{-1}.$$

Similar to the analysis of Modak [3] for 4-dimensional space-time one can interpret that the parameter K measures the $(D+1)$-space scalar curvature and γ_0 measures the deviation of the integral curves of the velocity vector field to be geodesics.

Another set of solutions can be obtained for $K=0$ i.e. for flat space-time. This is almost identical to those of Modak [3] for 4-dimensional space-time. So we shall not present it here.

Case II: When K is large i.e. $(D+1)$-dimensional spatial hypersurface has large curvature

Here we assume K to be infinitely large in such a manner that the metric coefficient e^{ω} is a finite (non-zero) quantity. Then (12) can be approximated by the equation (assuming $1/K$ to be very small so that second and higher powers are neglected)

$$\gamma''+\frac{\gamma'^2}{2}+\frac{3\gamma'}{r}-\frac{4\gamma'}{Kr^{\alpha+1}}+\frac{4(D-1)(2-\alpha)}{Kr^{\alpha+2}}=0. \tag{15}$$

To solve this equation, let us assume

$$\gamma=\bar{\gamma}+h, \tag{16}$$

where $\bar{\gamma}$ satisfies

$$\bar{\gamma}''+\frac{\bar{\gamma}'^2}{2}+\frac{3\bar{\gamma}'}{r}=0 \tag{17}$$

and the small correction h (assuming square and higher powers of the derivative of h are neglected) follows

$$h''+\frac{3h'}{r}=\frac{4\gamma'}{Kr^{\alpha+1}}-\frac{4(D-1)(2-\alpha)}{Kr^{\alpha+2}}, \tag{18}$$

Now, (17) can be solved easily to obtain

$$e^{\bar{\gamma}}=\left(1-\frac{1}{4cr^2}\right)^2, \quad c\neq 0$$

$$=\frac{1}{r^4}, \quad c=0 \tag{19}$$

where $c=c(t)$ is the integration constant. Hence from (18) one can write the integral form

$$h'=\frac{4}{Kr^3}\int\frac{r^{-\alpha+1}}{(cr^2-1/4)}\mathrm{d}r-\frac{4(D-1)}{Kr^{\alpha+1}}. \tag{20}$$

So γ can be expressed in close form only for $c = 0$, the expression is

$$\gamma = -4\ln r + \frac{16}{K}\left[1 + \frac{1}{\alpha(2-\alpha)}\right]r^{-\alpha}. \tag{21}$$

Case III: When the spatial curvature is very small (K is small so that K^2 and higher powers are neglected)

In this case, (12) reduces to

$$\gamma'' + \frac{\gamma'^2}{2} + \frac{\gamma'}{r}(4Kr^{\alpha} - 1) + 4K(n-1)(2-\alpha)r^{\alpha-2} = 0. \tag{22}$$

So proceeding to the previous case we get,

$$\gamma = \gamma_0 \ln(r^2 + c_0) + H \tag{23}$$

with

$$H' = 8K\gamma_0 r \int \frac{r^{\alpha-1}}{r^2 + c_0}\mathrm{d}r + 16\alpha K r^{\alpha-1}. \tag{24}$$

The only solution, which is in close form for $c_0 = 0$, is the following

$$\gamma = \gamma_0 \ln r^2 + 8K((\alpha/\alpha - 2)\cdot\gamma_0 + 2)r^{\alpha}. \tag{25}$$

One can add to this solution a second degree polynomial in r with arbitrary coefficients, which are functions of t alone.

3. Alternative solution by Glass and Bergmann

The condition of pressure isotropy in co-moving co-ordinates, following Glass [7] and Bergmann [1], gives rise to a second order partial differential equation

$$F\frac{\partial^2 A}{\partial x^2} + 2\frac{\partial F}{\partial x}\frac{\partial A}{\partial x} - \frac{A}{(D-1)}\frac{\partial^2 F}{\partial x^2} = 0. \tag{26}$$

Here $A = e^{\gamma/2}$, $F = e^{\omega/2}$ and $x = r^2$. This partial differential equation is very similar to that in the 4-dimensional case. So one can get solutions analogous to those of Bergmann [1], Modak [3], Maiti [2], Banerjee *et al* [4] and Deng [5]. So we shall not discuss these types of solutions here. Only we shall present here solutions which are different from those mentioned above.

Let us assume the following polynomial relation between the metric coefficients:

$$F = \mu A^n, \quad \mu \text{ and } n \text{ are arbitrary.} \tag{27}$$

Now substituting in (26) one obtains

$$A\frac{\partial^2 A}{\partial x^2} + \lambda\left(\frac{\partial A}{\partial x}\right)^2 = 0, \tag{28}$$

with

$$\lambda = \frac{n(2D-n-1)}{(D-n-1)}.$$

Hence the solution is

$$A = [a_0 r^2 + b_0]^{1/(\lambda+1)},$$
$$F = \mu[a_0 r^2 + b_0]^{n/(\lambda+1)}, \tag{29}$$

where $a_0 = a_0(t)$ and $b_0 = b_0(t)$ are arbitrary. This solution is different from the above mentioned solutions.

4. Discussion

In the present paper, we have presented cosmological solutions with heat flux in higher dimensions. Some of the solutions are analogous to those present in the literature for 4-dimensional case. Also we get a few solutions which are totally new. As the arbitrary parameter K is related to the spatial curvature, the solutions with K large and small correspond to the cases where the gravitational effect is large and small respectively. Also, in the case of the pressure isotropy condition we obtain a different solution from the standard solutions in 4-dimensions.

The higher dimensional study of heat flux does not show any special character compared to the 4-dimensional case. So we may conclude that the dimension of space-time has very little effect on heat flux.

Acknowledgement

The authors are thankful to the members of Relativity and Cosmology Research Centre, Department of Physics, Jadavpur University for helpful discussion.

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