PRAMANA
— journal of
physics

© Printed in India

Vol. 44, No. 4,
April 1995
pp. 317–322

Exact solutions of Einstein and Einstein-scalar equations in 2 + 1 dimensions

K S VIRBHADRA
Departamento de Física Teórica, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
Present address: Theoretical Astrophysics Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

MS received 18 October 1994

Abstract. A nonstatic and circularly symmetric exact solution of the Einstein equations (with a cosmological constant Λ and null fluid) in $2+1$ dimensions is given. This is a nonstatic generalization of the uncharged spinless BTZ metric. For $\Lambda = 0$, the spacetime is though not flat, the Kretschmann invariant vanishes. The energy, momentum, and power output for this metric are obtained. Further a static and circularly symmetric exact solution of the Einstein-massless scalar equations is given, which has a curvature singularity at $r = 0$ and the scalar field diverges at $r = 0$ as well as at infinity.

Keywords. Lower dimension gravity; BTZ metric; Einstein-scalar equations; energy-momentum pseudotensor.

PACS No. 04·20

In recent years there has been considerable interest in studying the Einstein theory of gravity in $2+1$ spacetime dimensions. Due to the smaller number of dimensions it has tremendous mathematical simplicity. The use of three dimensional gravity has been suggested as a test bed for the quantization of gravity [1]. The lowest spacetime dimension in which the Einstein theory of gravitation exists is three and it possesses some strange features. In three dimensions the number of independent components of the Riemann curvature tensor and the Einstein tensor are the same (six) and the Riemann tensor can be expressed in terms of the Einstein tensor. Consequently, the spacetime described by the vacuum solutions to the Einstein equations in $2+1$ dimensions are locally flat and there are no gravitational waves and no interaction between masses. However, in Newton's theory of gravitation in three dimensions a point mass produces an acceleration which falls off like $1/r$ [2]. Unlike the case in $3+1$ dimensions, the Einstein theory of gravitation has no Newtonian limit in $2+1$ dimensions. Therefore, the Einstein gravitational constant is undetermined in three dimensions.

Deser and Mazur [3] obtained the gravitational and Coulomb fields of a massive point charge in $2+1$ dimensional Einstein–Maxwell (EM) theory. As opposed to the case of $3+1$ dimensional EM theory, the Coulomb field behaves asymptotically as $1/r$ and the stress tensor is nonvanishing everywhere. The total energy diverges. The Kretschmann invariant diverges at $r = 0$ as well as at infinity [3]. There exists a static multi-charge solution in $3+1$ dimensional EM theory [4]. However, there are no static solutions corresponding to two or more charges in $2+1$ dimensions [3].

Arguing that the light emitted from the surface of a star will always escape to the infinity because there is no gravity outside matter (or energy), Giddings *et al* concluded that black holes do not exist in three dimensional Einstein theory of gravitation [5]. However, Bañados *et al* (BTZ) [6] have recently discovered a black hole solution to the EM equations (with a negative cosmological constant) in $2+1$ dimensions, which is characterized by mass, angular momentum, and charge parameters. The $2+1$ dimensional uncharged BTZ black hole solution has fascinated many physicists minds ([7] and references therein). Unlike the Kerr metric it is not asymptotically flat and there is no curvature singularity. However, it shares many features of its $3+1$ dimensional counterpart. For instance, it has an ergosphere, upper bound in angular momentum (for a given mass) and has similar thermodynamical properties [8].

In this paper we give two exact solutions: (*a*) a nonstatic and circularly symmetric solution of the Einstein equations with a cosmological constant and null fluid, which is a nonstatic generalization to the spinless uncharged BTZ metric and (*b*) a static and circularly symmetric solution of the Einstein-massless scalar equations. We obtain the energy, momentum, and power output for the nonstatic circularly symmetric metric (with zero cosmological constant) and compare the results with its $3+1$ dimensional counterpart. We use the geometrized units and follow the convention that Latin (Greek) indices take values 0 to 2 (1 to 2). x^0 stands for the time coordinate.

The Einstein field equations with a cosmological constant Λ and the null fluid present are

$$R_{ij}-\frac{1}{2}R\,g_{ij}+\Lambda g_{ij}=\kappa N_{ij}. \tag{1}$$

R_{ij} is the Ricci tensor and R is the Ricci scalar. κ is the Einstein gravitational constant. N_{ij} is the energy-momentum tensor due to the null fluid, which is given by

$$N_{ij}=V_i\,V_j, \tag{2}$$

where V_i is the null fluid current vector satisfying

$$V_i\,V^i=0. \tag{3}$$

A nonstatic and circularly symmetric exact solution of the above equations in $2+1$ dimensions is given, in coordinates $x^0=u, x^1=r, x^2=\varphi$, by the line element

$$\mathrm{d}s^2=-(-m(u)-\Lambda r^2)\mathrm{d}u^2-2\mathrm{d}u\,\mathrm{d}r+r^2\mathrm{d}\varphi^2. \tag{4}$$

The nonvanishing component of the energy-momentum tensor of the null fluid is

$$N_0^1=\frac{\dot{m}}{2\kappa r}. \tag{5}$$

The dot denotes the derivative with respect to the time coordinate u. The null fluid current vector is

$$V^i=g_1^i\sqrt{\frac{-\dot{m}}{2\kappa r}}. \tag{6}$$

The nonvanishing components of the Riemann curvature tensor are

$$R_{1220} = \Lambda r^2,$$
$$R_{1010} = -\Lambda,$$
$$R_{2020} = \frac{2m\Lambda r^2 + r(-\dot{m} + 2\Lambda^2 r^3)}{2}, \tag{7}$$

and the Kretschmann invariant is

$$R^{abcd}R_{abcd} = 12\Lambda^2. \tag{8}$$

The spacetime described by the line element (4) with $\Lambda = 0$ is though not flat, the Kretschmann invariant vanishes. When m is a constant, (4) gives the uncharged spinless BTZ metric.

Lindquist *et al* [9] obtained the energy, momentum and power output for the Vaidya metric. It is of interest to obtain the same for the circularly symmetric nonstatic metric (with $\Lambda = 0$) and compare the results with its $3+1$ dimensional counterpart. There are several prescriptions for calculating the energy, momentum and power output for a nonstatic general relativistic system. For example, one can use the Landau and Lifshitz (LL) pseudotensor [10]

$$L^{mn} = L^{nm} = \frac{1}{2\kappa} S^{mjnk}{}_{,jk}, \tag{9}$$

where

$$S^{mjnk} = -g(g^{mn}g^{jk} - g^{mk}g^{nj}). \tag{10}$$

S^{mjnk} has the symmetries of the Riemann tensor. The LL pseudotensor satisfies the local conservation laws:

$$\frac{\partial L^{mn}}{\partial x^m} = 0. \tag{11}$$

L^{00} and $L^{0\alpha}$ are, respectively, the energy density and momentum (energy current) density components. It is known that the LL pseudotensor gives the correct result if calculations are carried out in "Cartesian coordinates". Therefore, we transform the line element (4), with $\Lambda = 0$, to these coordinates, which is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + (1 + m(u))\left[\mathrm{d}t - \frac{x\mathrm{d}x + y\mathrm{d}y}{r}\right]^2. \tag{12}$$

The coordinates u, r, φ in (4) and t, x, y in (12) are related through

$$u = t - r,$$
$$x = r\cos\varphi,$$
$$y = r\sin\varphi. \tag{13}$$

The determinant $g \equiv |g_{ij}|$ is

$$g = -1 \tag{14}$$

and the contravariant components of the metric tensor are

$$
\begin{aligned}
g^{00} &= -m-2,\\
g^{11} &= \frac{-mx^2+y^2}{r^2},\\
g^{22} &= \frac{-my^2+x^2}{r^2},\\
g^{01} &= -\frac{x(1+m)}{r},\\
g^{02} &= -\frac{y(1+m)}{r},\\
g^{12} &= -\frac{xy(1+m)}{r^2}.
\end{aligned}
\tag{15}
$$

We are interested in calculating the energy and momentum density components and therefore the required components of S^{mjnk} are

$$
\begin{aligned}
S^{0101} &= \frac{-y^2(1+m)-r^2}{r^2},\\
S^{0202} &= \frac{-x^2(1+m)-r^2}{r^2},\\
S^{0102} &= \frac{xy(1+m)}{r^2},\\
S^{0112} &= \frac{y(1+m)}{r},\\
S^{0212} &= \frac{-x(1+m)}{r}.
\end{aligned}
\tag{16}
$$

Substituting (16) in (9), we get the energy and momentum (energy current) density components

$$
\begin{aligned}
L^{00} &= \frac{m'}{2\kappa r},\\
L^{01} &= -\frac{\dot{m}x}{2\kappa r^2},\\
L^{02} &= -\frac{\dot{m}y}{2\kappa r^2}
\end{aligned}
\tag{17}
$$

(m' stands for $\partial m/\partial r$). Therefore, the energy and momentum components are

$$
\begin{aligned}
E &= m\frac{\pi}{\kappa},\\
P_x &= P_y = 0,
\end{aligned}
\tag{18}
$$

and the power output is

$$W = -\frac{\pi}{\kappa}\frac{\mathrm{d}m}{\mathrm{d}u}. \tag{19}$$

For the Vaidya metric, Lindquist *et al* found $E = M, P_x = P_y = P_z = 0$, and $W = -\mathrm{d}M/\mathrm{d}u$, where M is the mass parameter in the Vaidya metric. Thus, we get similar results in its $2+1$ dimensional counterpart (κ is not known due to the lack of a Newtonian limit). This supports the importance of the study of lower dimension gravity as a toy model.

The exact solutions of the Einstein-massless scalar equations in $3+1$ spacetime dimensions are known in the literature [11]. It is of interest to obtain a static and circularly symmetric exact solution of these equations in $2+1$ dimensions. The Einstein-massless scalar field equations are [11]

$$R_{ij} - \frac{1}{2}R\,g_{ij} = \kappa\,S_{ij}, \tag{20}$$

where S_{ij}, the energy-momentum tensor of the massless scalar field, is given by

$$S_{ij} = \Phi_{,i}\Phi_{,j} - \frac{1}{2}g_{ij}g^{ab}\Phi_{,a}\Phi_{,b}, \tag{21}$$

and

$$\Phi_{,i}^{;i} = 0. \tag{22}$$

Φ stands for the massless scalar field.

A static and circularly symmetric exact solution of the above equations is given by the line element

$$\mathrm{d}s^2 = -B\,\mathrm{d}t^2 + B\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 \tag{23}$$

with

$$B = (1-q)\,\mathscr{R}^q, \tag{24}$$

and the scalar field

$$\Phi = \sqrt{\frac{q}{\kappa}}\ln \mathscr{R}, \tag{25}$$

where

$$\mathscr{R} = \frac{r}{r_0}. \tag{26}$$

q stands for the scalar charge. $q = 0$ gives the flat spacetime in $2+1$ dimensions. The nonvanishing components of the Einstein tensor and the energy-momentum tensor of the massless scalar field are given by

$$G_1^1 = -G_2^2 = -G_0^0 = \kappa S_1^1 = -\kappa S_2^2 = -\kappa S_0^0 = \frac{q}{2(1-q)\,\mathscr{R}^q r^2}. \tag{27}$$

The surviving independent components of the Riemann tensor are

$$R_{1212} = R_{2020} = \frac{q}{2},$$

$$R_{1010} = \frac{q(q-1)\mathscr{R}^q}{2r^2}, \tag{28}$$

and the Kretschmann invariant is

$$R^{abcd}R_{abcd} = \frac{3q^2}{(1-q)^2\,\mathscr{R}^{2q}r^4}. \tag{29}$$

The Kretschmann invariant diverges at $r = 0$ showing a curvature singularity there and it vanishes at infinity. However, the scalar field diverges at $r = 0$ as well as at infinity. The details of this paper will be given elsewhere. Upon completion of this work a preprint [12] came to our notice which has the nonstatic solution given here.

Acknowledgements

This work was supported in part by a Basque Government post-doctoral fellowship. Thanks are due to A Achúcarro, I Egusquiza, P S Joshi, and M A Valle for discussions.

References

[1] W G Unruh, *Int. J. Mod. Phys.* **D3**, 131 (1994)
E Witten, *Nucl. Phys.* **B311**, 46 (1988)
[2] J R Gott and M Alpert, *Gen. Relativ. Gravit.* **16**, 243 (1984)
[3] S Deser and P O Mazur, *Class. Quantum Gravit.* **2**, L51 (1985)
[4] A Papapetrou, *Proc. R. Irish Acad.* **51**, 91 (1947)
S D Majumdar, *Phys. Rev.* **72**, 390 (1947)
[5] S Giddings, J Abbott and K Kuchar, *Gen. Relativ. Gravit.* **16**, 751 (1984)
[6] M Bañados, C Teitelboim and J Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992)
[7] D Cangemi, M Leblanc and R B Mann, *Phys. Rev.* **D48**, 3606 (1993)
N Kaloper, *Phys. Rev.* **D48**, 2598 (1993)
A Achúcarro and M Ortiz, *Phys. Rev.* **D48**, 3600 (1993)
K Shiraishi and T Maki, *Class. Quantum Gravit.* **11**, 695 (1994)
A R Steif, *Phys. Rev.* **D49**, R585 (1994)
G Lifschytz and M Ortiz, *Phys. Rev.* **D49**, 1929 (1994)
S Hyun, G H Lee and J H Yee, *Phys. Lett.* **B322**, 182 (1994)
O B Zaslavskii, *Class. Quantum. Gravit.* **11**, L33 (1994)
[8] M Bañados, M Henneaux, C Teitelboim and J Zanelli, *Phys. Rev.* **D48**, 1506 (1993)
[9] R W Lindquist, R A Schwartz and C W Misner, *Phys. Rev.* **137**, B1364 (1965)
[10] L D Landau and E M Lifshitz, *The classical theory of fields* (Oxford, Pergamon Press, 1985) p. 280
[11] A I Janis, D C Robinson and J Winicour, *Phys. Rev.* **186**, 1729 (1969)
[12] J S F Chan, K C K Chan and R B Mann, *Interior structure of a charged spinning black hole in (2 + 1) dimensions* gr-qc/9406049 (1994)