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On singularity-free spacetimes

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Abstract. We consider here the metric for the singularity-free family of fluid models. The metric is unique for cylindrically symmetric space-time with metric potentials being separable functions of radial and time coordinates in the comoving coordinates. It turns out that fluid models separate out into two classes, with $\rho \neq \mu p$ in general but $\rho = 3p$ in particular and $p = \rho$. It is shown that in both the cases radial heat flow can be incorporated without disturbing the singularity-free character of the spacetime. The geodesics of the singularity-free metric are studied and the geodesic completeness is established. Several previously known solutions are derived as particular cases.

Keywords. General relativity; singularity-free models; exact solutions.

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1. Introduction

The standard view, the Universe had a big-bang singular origin predicted by the standard Friedman–Robertson–Walker (FRW) model, was supported on the formal grounds by the strong singularity theorems [1]. That occurrence of singularity is inescapable if we adhere to reasonable energy and causality conditions and to general relativity (GR). To handle the cosmic singularity several attempts have been made by several authors. They include modification of GR, invoking quantum effects and new fields, ascribing unusual properties to matter etc. By sacrificing energy conditions and other physical properties, it is easy to construct a singularity free model, the most well-known example of this class is the deSitter universe. That is there exists a large number of such models where acceptable physical behaviour of matter has been traded off for avoidance of singularity.

Under this background Senovilla's singularity-free solution [2] is remarkable that it satisfies the energy conditions and has very acceptable equation of state $\rho = 3p$. This is the first solution of its kind that is free of singularity with physically reasonable behaviour of matter. Further Ruiz and Senovilla [3] have discussed the singularity-free character of inhomogeneous cylindrically symmetric spacetime and have separated out a family of singularity-free perfect fluid models. It has been shown [4] that these solutions are geodetically complete and are able to escape the singularity theorems because they do not obey the assumption of existence of causal compact trapped surfaces.

From the point of view of general study of singularity-free cosmology it would be of relevance to examine how robust is the singularity-free character of the spacetime? That is, can we generate singularity-free vacuum solutions or can we introduce heat flow, viscosity etc. retaining the singularity-free character? In this context we have [5] considered viscous fluid and found that it is not possible to have viscosity non-negative for all times while heat flow [6] can be incorporated easily. We give the heat flow generalizations of the two classes of fluid models mentioned above.

In § 2 we obtain the singularity-free metric. We also consider the regularity of Weyl and Ricci curvatures for this metric. Section 3 is devoted to the study of perfect fluid models and the fluid models with radial heat flow. In § 4 we present a general study of geodesics of the singularity-free metric. We also explicitly exhibit the non-occurrence of compact trapped surfaces that makes the singularity theorems inapplicable to this family of metrics. We conclude in § 5 with discussion of the results obtained in the paper.

2. Singularity-free metric

Ruiz and Senovilla [3] have separated out a singularity-free family of solutions of Einstein's equations for perfect fluid. The spacetime has cylindrical symmetry. With metric potentials as separable functions of r and t the family is unique.

We begin with the general form of the metric

$$ds^2 = A^2(dt^2 - dr^2) - B^2 dz^2 - C^2 d\phi^2. \tag{1}$$

Here we have $A = A_1(t)A_2(r), B = B_1(t)B_2(r), C = C_1(t)C_2(r)$ and have used the coordinate freedom to write $g_{tt} = |g_{rr}|$. The metric admits two space-like Killing vectors $\partial/\partial z$ and $\partial/\partial\phi$ which are mutually as well as hypersurface orthogonal implying cylindrical symmetry. That means we should write $C_2(r) = rf(r)$ so as to satisfy the regularity conditions on the axis $r = 0$ and the elementary flatness condition in its vicinity.

The prime requirement for a metric to be singularity-free is that its coefficients have no zeroes for real values of the co-ordinates and Ricci and Weyl tensors are regular everywhere. Ricci curvatures for the metric (1) are given in the Appendix and they will be regular provided $\dot{A}/A$, A'/A etc. are regular. At the outset $C_2(r) = rf(r)$ would imply $A_1(t) = C_1(t)$ (by demanding: coefficient of $r^{-1} = 0$). This regularity condition has no reference to matter distribution.

Assuming the matter content of the spacetime to be perfect fluid we impose the fluid conditions, $R_{14} = 0$, and $R_{11} = R_{22} = R_{33}$ for the comoving velocity field

$$\mathbf{u} = A dt. \tag{2}$$

After some calculations it can be shown that $R_{14} = 0$ implies that $A_1(t) = C_1(t) = T^{\alpha}(t)$, $B_1(t) = T^{\beta}(t)$ and $A_2(r) = B_2^q(r)$. That is the time variation in the metric stems from one singile function. Subsequently $R_{11} = R_{22} = R_{33}$ (from Appendix) will imply

$$\frac{\ddot{T}}{T} + (\alpha + \beta - 1)\frac{\dot{T}^2}{T^2} = k^2, \tag{3}$$

$$\frac{C_2''}{C_2} - \frac{B_2''}{B_2} = (\alpha - \beta)k^2, \tag{4}$$

$$C_2 = B_2' B_2^{(\alpha-\beta)/\beta}. \tag{5}$$

Equation (3) admits the general singularity-free solution

$$T(t) = K_1[C(\lambda kt) + t_0)]^{1/\lambda^2} = C^{1/\lambda^2}(\lambda kt), \tag{6}$$

where we have chosen the integration constants $K_1 = 1$, $t_0 = 0$ and written $\alpha + \beta = \lambda^2$, $C(x) = \cosh(x)$. The other solutions for $\lambda^2 \leqslant 0$ have been ruled out by the singularity-free and positivity of density conditions.

In [3] it was assumed that $\alpha + \beta = 1$ and the time dependence stemming from the same function $T(t)$ which we have shown above has to follow from $R_{14} = 0$. It turns out that there is no loss of generality in the former assumption either, for it would just imply an overall scaling in the expressions for density and pressure (i.e. $C^{\alpha}(kt) \to C^{\alpha/\lambda^2}(\lambda kt)$). We can hence take $\alpha + \beta = \lambda^2 = 1$. For space-dependance Ruiz and Senovilla employ four arbitrary functions ($g_{44} \neq |g_{11}|$) and obtain the general solution of the system of equations (4) and (5) with the condition $\alpha + \beta = 1$. The solution in our notation, will read as

$$MX' = C_2/B_2,\ B_2 = X^{(1-\alpha)/(2\alpha-1)}$$

$$C_2^2 = B_2^2[(2\alpha-1)^2 k^2 M^2 X^2 + NX^{(4\alpha-3)/(2\alpha-1)} - L]$$

where $X = X(r)$ and M, N, L are constants of integration. This family is cylindrically symmetric and contains a sub-family which is non-singular. This family can be shown to be unique for the general orthogonal metric separable in space and time variables [7]. In Ruiz-Senovilla case $X(r)$ is arbitrary which gets determined for us because of the choice $g_{44} = |g_{11}|$. It turns out that for non-singular family we can integrate for X in terms of elementary regular functions, only for $N = 0$ to give the general solution

$$B_2 = C^b(mr),\ C_2 = m^{-1}S(mr)C^c(mr)$$

where $S(mr) = \sinh(mr)$ and $A_2 = B_2^q = C^a(mr)$. The fluid consistency equations (4), (5) and the equation $R_{14} = 0$ will give relations between the parameters a, b, c, m, k and α.

Thus we obtain the metric for singularity free space-time as

$$\begin{aligned} ds^2 = {} & C^{2\alpha}(kt)C^{2a}(mr)(dt^2 - dr^2) - C^{2\beta}(kt)C^{2b}(mr)dz^2 \\ & - m^{-2}S^2(mr)C^{2c}(mr)C^{2\alpha}(kt)d\phi^2 \end{aligned} \tag{7}$$

which is unique for perfect fluid distribution with appropriate relations between free parameters.

If we do not use the co-ordinate freedom to write $g_{44} = |g_{11}|$, then the most general unique singularity-free metric is [3]

$$\begin{aligned} ds^2 = {} & C^{1+n}(kt)C^{n-1}(nkr)\left[dt^2 - \frac{S^2(nkr)}{P^2}dr^2\right] - C^{1-n}(kt)C^{(n-1)/n}(nkr)dz^2 \\ & - C^{1+n}(kt)\frac{P^2}{n^2k^2L^2C^{(n-1)/n}(nkr)}d\phi^2 \end{aligned} \tag{8}$$

where $L = M - (M-1)/2n$, $P^2 = C^2(nkr) + (M-1)C^{(2n-1)/n}(nkr) - M$; n, k and M being constants. The metric (8) satisfies Einstein field equations for a perfect fluid distribution. The metric (7) follows from (8) when $M = 1$. We [8] have shown elsewhere that the metric (7) can be deduced through a natural anisotropization and inhomogeneisation of the FRW metric with negative curvature. We shall not give these details here.

The kinematic parameters expansion θ, shear σ^2 and acceleration $\dot{u}_r$ for the metric (7) are given by

$$\theta = (\alpha + 1)kS(kt)C^{-\alpha-1}(kt)C^{-a}(mr) \tag{9}$$

$$\sigma^2 = \frac{2}{3}(2\alpha - 1)^2 k^2 S^2(kt)C^{-2(\alpha+1)}(kt)C^{-2a}(mr) \tag{10}$$

$$\dot{u}_r = -amS(mr)C^{-\alpha}(kt)C^{-a-1}(mr). \tag{11}$$

They are all regular and finite throughout the spacetime. The Ricci and Weyl curvatures will also be regular as they involve $\dot{A}/A$, A'/A etc. which are all regular and finite for the metric (7). This would also imply the regular behaviour for the physical parameters ρ and p. There would however be some conditions on the parameters.

It is the non-vanishing of acceleration and shear that is responsible for avoidance of singularity. It is physically conceivable that acceleration and shear do not let the fluid congruence to focus into small enough a region to form trapped surfaces leading to singularity. Acceleration of congruence viewed as spatial *pressure gradient* which opposes gravitational attraction and provides the bounce to transform contraction into expansion at $t = 0$. For the metric (1) acceleration can be non-zero only if shear is non-zero [7]. The shear makes geodesics of the congruence to slip through without letting them converge into a small region. It helps in defocussing of the congruence. It is however obvious that their presence alone is not sufficient to avoid singularity as we can easily check by letting $C(kt) \to S(kt)$ in (7). Then the spacetime is singular at $t = 0$ with acceleration and shear non-zero. Hence it may be a necessary condition but not sufficient. For sufficiency we should have regularity of the metric and curvatures. Then energy conditions will have to be satisfied.

For the metric (7) we give below the Ricci and Weyl curvatures to show explicitly that they are regular. The expressions read as follows:

$$2(\psi_0 + \psi_4)A^2 = m^2(2a - 3c + b - 1) - k^2(\alpha - \beta) + 2\alpha(\alpha - \beta)k^2\, T^2(kt) + m^2[2a(c - b) + (b - c)(b + c - 1)]\, T^2(mr) \tag{12}$$

$$2(\psi_0 - \psi_4)A^2 = mk(\alpha + \beta - 1)\, T(kt)\, T^{-1}(mr) + \{2\alpha(c - b) + \alpha(2a - b - c) + \beta(b + c - 2a) + (b - c)\}\, T(mr) \tag{13}$$

$$2\psi_2 A^2 = k^2(1 - 2\alpha) + m^2(2a + b - 3c - 1) + (b + c - 2a + 2bc - b^2 - c^2)m^2\, T^2(mr) + (2\alpha + \alpha^2 + \beta^2 - 2\alpha\beta - 1)k^2\, T^2(kt) \tag{14}$$

$$A^2 R_{11} = m^2(1 + b + 3c) - \alpha k^2 + \{b(b - 1) + c(c - 1) - a(b + c + 1)\}m^2\, T^2(mr) - \{\alpha(\alpha + \beta - 1)\}k^2\, T^2(kt) \tag{15}$$

$$A^2 R_{22} = 2bm^2 - \beta k^2 - \beta(\alpha + \beta - 1)k^2\, T^2(kt) + \{b(b - 1) + bc\}m^2\, T^2(mr) \tag{16}$$

$$A^2 R_{33} = m^2(b + 3c + 1) - \alpha k^2 - \alpha(\alpha + \beta - 1)k^2 \, T^2(kt) + \{c(c-1) + bc\} m^2 \, T^2(mr) \tag{17}$$

$$A^2 R_{44} = (2\alpha + \beta)k^2 - 2am^2 + a(1 - b - c)m^2 \, T^2(mr) + \{\beta(\beta - 1) + \alpha(\alpha - 1) - \alpha(\alpha + \beta + 1)\} k^2 \, T^2(kt) \tag{18}$$

$$A^2 R_{14} = mk[\beta(b - a) - \alpha(a + b)] \, T(kt) \, T(mr) \tag{19}$$

where

$$A^2 = C^{2\alpha}(kt) C^{2a}(mr), \; T(x) = \tanh(x). \tag{20}$$

The above quantities will be singularity-free if the coefficients of $T^{-1}(mr)$ vanish, so we obtain the condition

$$\alpha + \beta = 1. \tag{21}$$

With α, β as restrained above, Ricci and Weyl curvatures are regular and finite everywhere indicating that the metric is free of any kind of singularity. Now we can introduce matter to determine other parameters occurring in R_{ik} consistent with the energy conditions.

The conditions for perfect fluid $R_{14} = 0$ and $R_{11} = R_{22} = R_{33}$ would imply

$$(\alpha - \beta)(\alpha + \beta - 1) = 0 \tag{22}$$

$$(\beta - \alpha)k^2 = (b - 1 - 3c)m^2 \tag{23}$$

$$(b - c)(b + c - 1) = 0 \tag{24}$$

$$a(b + c + 1) = b(b - 1 - c) \tag{25}$$

$$\alpha(a + b) = \beta(b - a). \tag{26}$$

The first condition is satisfied in view of (21) which has to be obeyed always as it is dictated by the regularity of Weyl tensor.

The remaining conditions give rise to the following two cases (i) $b = c$, (ii) $b + c = 1$. These are the only two possibilities for singularity-free fluid models satisfying the energy conditions. The former in general represents a fluid without an equation of state. Only when $b = -1/3$, it represents Senovilla's [2] radiation model with $\rho = 3p$. The latter will always have $\rho = p$, representing stiff fluid model [9]. In this case when we put $\rho = 0$, we get two distinct vacuum solutions as its matter free limit. The point to be noted is that these are all general unique solutions for the singularity-free form (7). It is also interesting that the two cases $\rho \neq \mu p$ in general (but $\rho = 3p$ in particular) and $\rho = p$ separate out nicely.

3. Fluid models

We shall consider the perfect fluid and fluid with radial heat flux models. It is possible to introduce heat flux in both the cases considered above.

3.1 *Perfect fluid*

Einstein's equations for non-empty spacetime are

$$R_{ik} = -8\pi(T_{ik} - \tfrac{1}{2}Tg_{ik}) \tag{27}$$

where for perfect fluid

$$T_{ik} = (p + \rho)u_i u_k - pg_{ik}, \quad u_i u^i = 1. \tag{28}$$

Case (i) $b = c$

We write from eqs (23)–(26)

$$\alpha = (1 + b)(1 + 2b)^{-1}, \quad a = -b(1 + 2b)^{-1}, \quad \frac{m}{k} = n = (1 + 2b)^{-1}. \tag{29}$$

This determines the metric (7) in terms of the two free parameters b and k, say.
The pressure and density have the expressions:

$$8\pi pA^2 = \frac{b^2(1-2b)}{(1+2b)^3}k^2 C^{-2}(nkr) - \frac{(3b+1)(b+1)}{(1+2b)^2}k^2 C^{-2}(kt) \tag{30}$$

$$8\pi\rho A^2 = \frac{b(2b-1)(2+3b)}{(1+2b)^3}k^2 C^{-2}(nkr) - \frac{(3b+1)(b+1)}{(1+2b)^2}k^2 C^{-2}(kt) \tag{31}$$

where $A = C^{\alpha}(kt)C^{a}(nkr)$. This is the Ruiz-Senovilla [3] model. For satisfaction of the energy conditions we shall require $-\frac{1}{2} < b \leqslant -\frac{1}{3}$ and the Senovilla radiation model results for $b = -\frac{1}{3}$. With b so restricted, ρ and p are positive for entire spacetime. For $b \neq -\frac{1}{3}$, there is no equation of state of the type $\rho = \mu p$. The maximum density would occur at $t = 0$ and $r = 0$ which decreases to zero as $t \to \pm\infty$ and $r \to \infty$. The parameter k can be identified with $\rho_{\max}$ which can be freely chosen. At a given t, ρ is largest at the origin $r = 0$ and at given r, it is largest at $t = 0$.

Case (ii) $b + c = 1$

In this case we have

$$\alpha = \tfrac{1}{2}(2 - b), \quad a = b(b - 1), \quad c = 1 - b, \quad n^2 = \frac{m^2}{k^2} = \frac{1}{4} \tag{32}$$

and we get the stiff fluid [5] model with

$$8\pi\rho = 8\pi p = \left(\frac{b^2 - 4}{4A^2}\right)k^2 C^{-2}(kt). \tag{33}$$

Clearly the energy condition will be obeyed for $b^2 \geqslant 4$, i.e. b lying outside the interval $-2 \leqslant b \leqslant 2$, the end points of which give the two distinct vacuum cosmological solutions [9]. These solutions are given as follows:
For $b = 2$

$$ds^2 = C^4(mr)(dt^2 - dr^2 - C^2(2mt)dz^2) - m^{-2}S^2(mr)C^{-2}(mr)d\phi^2. \tag{34}$$

For $b = -2$

$$ds^2 = C^4(2mt)[C^{12}(mr)(dt^2 - dr^2) - C^6(mr)m^{-2}S^2(mr)d\phi^2] - C^{-2}(2mt)C^{-4}(mr)dz^2. \tag{35}$$

It can be easily verified from eqs (12)–(14) that both the solutions are Petrov type I. The former (34) is static as Weyl scalars have no time dependence, while the latter can asymptotically represent a plane gravitational wave. Since there are no obvious localized sources for the solutions, hence the only possible source can be gravitational radiation. However, one knows that in realistic situation the metric for gravitational radiation is of type I almost everywhere [10].

3.2 *Fluid with heat flow*

The energy momentum tensor for fluid with heat flow is given by

$$T_{ik} = (p + \rho)u_i u_k - p g_{ik} + (q_i u_k + q_k u_i), \quad u_i u^i = 1, \quad u_i q^i = 0. \tag{36}$$

Here q_i is the heat flow vector. Here we are relaxing the condition (26) by taking radial heat flow. We take the tetrad components of q_i as $q_a = (q, 0, 0, 0)$. The parameters ρ, p and q are given by

$$8\pi\rho A^2 = (a - 3b)m^2 + (\beta - \alpha)k^2 + (\beta^2 + 2\alpha\beta + \alpha - \beta)k^2 T^2(kt) + \tfrac{1}{2}(a - 3b)(b + c - 1)m^2 T^2(mr) \tag{37}$$

$$8\pi p A^2 = (a + b)m^2 - (\alpha + \beta)k^2 + (\alpha + \beta - \beta^2)k^2 T^2(kt) + \tfrac{1}{2}(a + b)(b + c - 1)m^2 T^2(mr) \tag{38}$$

$$8\pi q A^2 = mk[\alpha(a + b) - \beta(b - a)]\ T(kt)\ T(mr) \tag{39}$$

where $A = C^\alpha(kt)C^a(mr)$.

Case (*i*) $b = c$

In this case we have

$$b = c, \alpha = 1 - \beta, a = -\frac{b}{1 + 2b}, n^2 = \frac{m^2}{k^2} = \frac{1 - 2\beta}{1 + 2b}. \tag{40}$$

We treat b and n as arbitrary parameters. The physical parameters are then given by

$$8\pi q A^2 = \frac{k^2 nb}{(1 + 2b)}\{(n^2 - 1) + 4b(b + 1)n^2\}\ T(kt)\ T(nkr) \tag{41}$$

$$8\pi p A^2 = \frac{k^2}{4}(n^2 - 1)\{1 - n^2(1 + 2b)^2\} + \frac{b^2(1 - 2b)}{(1 + 2b)}n^2 k^2 C^{-2}(nkr) + \tfrac{1}{4}\{n^2(1 + 2b) - 3\}\{1 + n^2(1 + 2b)\}C^{-2}(kt) \tag{42}$$

$$8\pi\rho A^2 = \frac{k^2}{4}(3+n^2)\{1-n^2(1+2b)^2\}$$

$$-\frac{b(1-2b)(2+3b)}{(1+2b)}n^2k^2C^{-2}(nkr)$$

$$+\tfrac{1}{4}\{n^2(1+2b)-3\}\{1+n^2(1+2b)\}C^{-2}(kt). \tag{43}$$

From the above equations it can be easily seen that the physical requirements $p \geqslant 0, \rho \geqslant 0$ are satisfied for

$$-\frac{1}{2} < b \leqslant -\frac{1}{3}, \quad \frac{3}{1+2b} < n^2 \leqslant \frac{1}{(1+2b)^2}. \tag{44}$$

From (41) it is clear that $q = 0$ implies $n^2 = (1+2b)^{-2}$ and the model reduces to the perfect fluid (29). For $n = 3$ we have the heat flow generalization of the Senovilla radiation model [6].

Case (ii) $b + c = 1$

Here we write

$$a = b(b-1), \quad C = 1-b, \quad \alpha = 1-\beta, \quad n^2 = \frac{m^2}{k^2} = \frac{2\beta-1}{4(b-1)} \tag{45}$$

and obtain

$$8\pi q A^2 = b(b-1)(4n^2-1)nk^2\, T(kt)\, T(nkr) \tag{46}$$

$$8\pi p A^2 = \frac{k^2}{4}(4n^2-1)\{1-4n^2(b-1)^2\}$$

$$+\frac{k^2}{4}[4n^2(b-1)-1][4n^2(b-1)+3]C^{-2}(kt) \tag{47}$$

$$8\pi\rho A^2 = -\frac{k^2}{4}[4n^2-1][3+4n^2(b-1)^2]$$

$$+\frac{k^2}{4}[4n^2(b-1)-1][4n^2(b-1)+3]C^{-2}(kt). \tag{48}$$

The physical requirements $\rho > 0$ and $p \geqslant 0$ restrict n and b to the ranges

$$0 < n^2 \leqslant \tfrac{1}{4}, \quad 1 < b < 2. \tag{49}$$

For $n^2 = \frac{1}{4}$, it reduces to the stiff fluid model (32). The phenomenological expression for the heat conduction is given by

$$q_k = \psi\left(\frac{\partial F}{\partial x^i} + F\dot{u}_i\right)(\delta^i_k - k^i u_k) \tag{50}$$

where ψ is the thermal conductivity and F is the temperature. For the cases under consideration only radial flow is retained. Hence eq. (50) can be integrated if ψ is a function of t alone to give

$$F = l(t)C^{a}(nkr) - \frac{k}{16\pi a\psi}\{\alpha(a+b) - \beta(b-a)\}C^{-a}(nkr)C^{-\alpha}(kt)\,T(kt) \tag{51}$$

where $l(t)$ is an arbitrary function of t. Thus temperature distributions for the cases (i) and (ii) can be obtained.

The heat flux has been considered in the evolution of the cosmological models by several authors (see [6]). Here it should be noted that the metric (7) has been considered by Tikekar *et al* [11] in connection with inhomogeneous string models.

4. Geodesics

To demonstrate the geodesic completeness of the metric, we should show that all non-spacelike geodesics can be extended to arbitrary values of the affine parameter. For this let us write the geodesics of the metric (7) following the standard procedure and they would read as follows:

$$C^{2\alpha}(kt)C^{2a}(mr)(\dot{t}^2 - \dot{r}^2) - L^2 C^{-2\beta}(kt)C^{-2b}(mr) - m^2 M^2 S^{-2}(mr)C^{-2\gamma}(kt)C^{-2c}(mr) = \delta, \tag{52}$$

$$C^{2\beta}(kt)C^{2b}(mr)\dot{z} = L, \tag{53}$$

$$m^{-2}S^2(mr)C^{2\gamma}(kt)C^{2c}(mr)\dot{\phi} = M, \tag{54}$$

$$\ddot{r} + amT(mr)(\dot{t}^2 + \dot{r}^2) + 2\alpha kT(kt)\dot{t}\dot{r} - M^2 m^3 S^{-3}(mr)C^{-2(\gamma+\alpha)}(kt)C^{-2(c+a)+1}(mr)(1 + cT^2(mr)) - L^2 bmS(mr)C^{-2(\beta+\alpha)}(kt)C^{-2(b+a)-1}(mr) = 0, \tag{55}$$

$$\ddot{t} + \alpha kT(kt)(\dot{t}^2 + \dot{r}^2) + 2amT(mr)\dot{t}\dot{r} + \gamma km^2 M^2 S^{-2}(mr)S(kt)C^{-2(\gamma+\alpha)-1}(kt)C^{-2(c+a)}(mr) + \beta kL^2 S(kt)C^{-2(\beta+\alpha)-1}(kt)C^{-2(b+a)}(mr) = 0. \tag{56}$$

A dot denotes derivative with respect to the affine parameter, and L and M are the two constants of motion corresponding to the two Killing vectors representing the conserved z and ϕ momenta. The another constant of motion is δ due to the rest mass which is one for timelike and zero for null geodesics. Since all the terms in (52)–(56) are non-singular hence the solutions to the equations will exist and they will be unique.

We shall now examine the behaviour of first and second derivatives of the coordinates relative to the affine parameter. We should put finite bounds on the first derivatives [12] which will imply that the geodesics are complete. However, the second derivatives should not be singular to ensure that the field is overall non-singular. Without any loss of generality we can restrict to the future pointing geodesics. In this discussion, we do not restrict to fluid distributions, i.e. the parameters $\alpha, \beta, \gamma, a, b, c, m, k$ are treated as arbitrary. Following [4], we shall first consider the particular geodesics and then come to the general case.

(*a*) *Fluid congruence*: The simplest geodesics are the ones associated with the fluid congruence. For these we have $\dot{r} = \dot{\phi} = \dot{z} = 0$. It follows from (2) that the only geodesic possible in this case has to lie on the axis $r = 0$ and it is given by

$$\dot{t} = C^{-\alpha}(kt) \leqslant 1 \quad \text{for} \quad \alpha \geqslant 0.$$

That means α must be non-negative and then the geodesics are obviously complete.

(*b*) *Axial geodesics*: For the geodesics lying on the axis we have $\dot{r} = 0 = \dot{\phi}, r = 0$ and so we write

$$\dot{z} = LC^{-2\beta}(kt),$$

$$\dot{t} = C^{-\alpha}(kt)[L^2 C^{-2\beta}(kt) + \delta]^{1/2} \leqslant (L^2 + \delta)^{1/2},$$

provided $\alpha + \beta \geqslant 0$. Then the derivatives will be bounded everywhere and the geodesics will be complete. Note that the coordinate t attains infinite value only when so does the affine parameter. This property will be shared by all the geodesics and will not be referred to henceforth.

(*c*) *Radial null geodesics*: Here $\dot{z} = 0 = \dot{\phi}, \delta = 0$, the first integrals give

$$\dot{t} = |\dot{r}|, \quad \dot{t} = hC^{-2\alpha}(kt)C^{-2a}(mr), \quad h = \text{constant}.$$

It is clear that $|\dot{r}| = \dot{t} \leqslant h$ provided $\alpha \geqslant 0, a \geqslant 0$ and then the geodesics will be complete. The geodesics with fixed ϕ comfortably continue along $\pi + \phi$ after crossing the axis.

(*d*) *Radial timelike geodesics*: Here $\delta = 1$, $\dot{z} = 0 = \dot{\phi}$. Let us parametrize $\dot{t}$ and $\dot{r}$ by writing

$$t = C(v)A(r,t), \quad \dot{r} = S(v)A(r,t), \dot{v} = B(r,t,v)A(r,t)$$

where $A = C^{-\alpha}(kt)C^{-a}(mr)$. Then

$$B = -[amT(mr)C(v) + \alpha kT(kt)S(v)].$$

We have $a \geqslant 0, \alpha \geqslant 0$ and take $k, m \geqslant 0$ which will imply $\dot{v} \leqslant 0$. We shall consider the role of the second derivatives in the end. The same reasoning will apply here to show that geodesics are complete.

(*e*) *Null geodesics with zero angular momentum*: In this case we have $\dot{\phi} = 0, \delta = 0$ and as before we write

$$\dot{t} = C(v)E(r,t), \dot{r} = S(v)E(r,t),$$

and

$$\dot{v} = E(r,t)F(r,t,v).$$

Then we obtain

$$E = |L|C^{-(\alpha+\beta)}(kt)C^{-(a+b)}(mr),$$

$$F = -[k(\alpha - \beta)\, T(kt)S(v) + m(a - b)\, T(mr)C(v)]$$

We shall require $\alpha + \beta \geqslant 0, a + b \geqslant 0, \alpha \geqslant \beta$ and $a \geqslant b$ for $\dot{v} \leqslant 0$.

(*f*) *Null geodesics on the hypersurfaces $z = constant$*: These are defined by $\dot{z} = 0, \delta = 0$, and as before we have

$$\dot{t} = C(v)P(r,t), \dot{r} = S(v)P(r,t)$$

and

$$\dot{v} = P(r,t)D(r,v)$$

where

$$P = |M| m S^{-1}(mr) C^{-(\gamma+\alpha)}(kt) C^{-(a+c)}(mr),$$

$$D = k(\gamma - \alpha)\, T(kt) S(v) + mC(v)[(c-a)\, T(mr) + T^{-1}(mr)].$$

As has been noted earlier that the regularity of the Ricci tensor requires $\alpha = \gamma$. Hence $D = D(r,v)$ as assumed.

In this case we can obtain one of the equations of the orbit that on integration yields

$$C(v) = N^{-1} S(mr) C^{c-a}(mr), \quad N = \text{constant} > 0.$$

Since $C(v) \geqslant 1$, the coordinate r is bounded between the roots of $S(mr)C^{c-a}(mr) = N$. Clearly $|v|$ is bounded and the geodesics are complete. Among them there could be a circular geodesic, corresponding to the double root, which will go around the axis at the fixed radius.

(*g*) *General non-spacelike geodesics*: Now we come to the general case. From (55) it follows that $\ddot{r}$ will be negative for positive t and increasing r provided $b \leqslant 0$ (for large r the term with L^2 will dominate over that with M^2). With $\ddot{r} < 0$, the r-coordinate cannot diverge to infinity in a finite proper time. As r decreases the $\dot{t}\dot{r}$ terms will dominate over the $(\dot{t}^2 + \dot{r}^2)$ term while the L^2 term will tend to zero as $r \to 0$. In the vicinity of the axis $r = 0, \ddot{r} > 0$ for decreasing r and $t > 0$. Thus the geodesics cannot collapse quickly enough into the axis to become singular. It is this feature that really provides the *bounce off* to the universe, turning contraction to expansion at $t = 0$.

The above arguments will apply to the cases (d) and (e) above and hence the geodesics will be complete for them. As regards the t-coordinate, $\ddot{t}$ should be negative for large values of t, so that it keeps its growth in check, not letting it diverge to infinity for finite values of the affine parameter. From (56), it can be seen that this will be so even if $\beta \leqslant 0$ because the $(\dot{t}^2 + \dot{r}^2)$ will always be the dominant term (for decreasing r the $\dot{t}\dot{r}$ term will not be relevant). $\dot{z}$ is always regular so long as t and r are regular. $\dot{\phi}$ diverges as $r \to 0$ but with $\dot{\phi} \neq 0$, the geodesics can never reach $r = 0$ as has been demonstrated in (f) above. In the neighbourhood of the axis eq. (54) simply represents the centrifugal effect.

It is straightforward to carry out the similar calculations as done in the previous cases but they are very cumbersome and not very enlightening and hence we will not report them here. It can be verified that the general geodesics are also complete. Let us now collect together all the restrictions prescribed by the above considerations on the parameters and they are

$$\alpha \geqslant 0, \alpha + \beta \geqslant 0, \alpha \geqslant \beta$$

$$a \geqslant 0, a + b \geqslant 0, a \geqslant b, b \leqslant 0, k \geqslant 0, m \geqslant 0$$

and $\alpha = \gamma$ implied by the regularity of the Ricci tensor. It is interesting to note that α, β and a, b satisfy the similar conditions. All this follows purely by requiring the geodesics to be complete without any reference to the matter distribution. That is in this framework one can *a priori* ensure the geodesic completeness by adhering to the above conditions on the parameters. The singularity-free perfect fluid models discussed above satisfy all these restrictions.

5. Discussion

If one were to write a singularity-free metric, without reference to anything else, the natural choice for metric functions would have been hyperbolic or quadratic functions without zeros. The amazing thing is that this obvious choice is the right and the only choice. However, one has to take cylindrical symmetry in place of the usual spherical symmetry.

For the avoidance of singularity acceleration and shear play the crucial role. In our case, the former cannot exist without the latter [7]. The former provides the bounce to the universe at $t = 0$ where contraction turns into expansion while the latter makes the fluid geodesic congruence to slip through without letting them to focus in a small enough region to form compact surfaces leading to singularity. The overall scenario is: universe has very low density tending to zero at $t \to -\infty$, wherefrom it starts contracting and attains the dense state at $t = 0$ where contraction changes to expansion and it expands to low density again at $t \to \infty$. The momentum gained during the contraction phase takes the universe through $t = 0$ to the expanding phase. The maximum density at $t = 0$ and $r = 0$ can be made as large or small as one pleases by choosing the parameter k. All physical and kinematic parameters remain regular and finite throughout the universe. The presence of acceleration and shear may be necessary but by no means sufficient to avoid singularity. In addition the spacetime should be regular.

It follows from the analysis of §4 that all the conclusions drawn in [4] for the geodesics of radiation model [2] remain true for our general case as well. We shall now demonstrate that the space-time does not admit trapped surfaces.

For this we should compute trace of the two null second fundamental forms [1] and that is given by

$$(g^{ab} X_{ab}) = \frac{1}{\sqrt{2}} C^{-\alpha}(kt) C^{-a+1}(mr) S^{-1}(mr) \times$$
$$[\mp m \mp m(b+c)\, T^2(mr) - k(\gamma + \beta)\, T(kt)\, T(mr)].$$

It is clear that $(X^a_a)_- \leqslant 0$ provided $b + c - 1 \geqslant 0$ and $(X^a_a)_+ \geqslant 0$ for $m(b + c + 1) \geqslant k(\alpha + \beta)$. Since the traces have opposite signs, there exist no closed trapped surfaces. This means that the outgoing and incoming radial null geodesics are respectively expanding and contracting everywhere. For existence of a closed trapped surface, in some region they should all be contracting, i.e. the trace should be positive for both in some region. It can be verified that the singularity-free models for §3 satisfy the conditions just deduced. Thus there occur no closed trapped surface in the singularity-free family of metrics.

Let us recall that $\alpha = \gamma$ was dictated by the cylindrical symmetry while $\alpha + \beta = 1$ for the metric (7) is demanded by the Weyl regularity. These two are general conditions without reference to any matter field. That is of the three α, β, γ only one is free for the singularity-free spacetime. It may be noted that all the fluid models (which are the only ones for the metric (7)) discussed in §3 obey the conditions, $\alpha \geqslant 0, \alpha + \beta \geqslant 0, \alpha \geqslant \beta, a \geqslant 0, a + b \geqslant 0, a \geqslant b$ and $b \leqslant 0$, obtained for completeness of geodesics.

The most important question for the singularity-free models is how to evolve them into the standard FRW model which successfully describes the present day universe. The affirmative answer to this question will bring these models into the active arena of cosmology and would perhaps have very significant role to play in the early universe cosmology. The main difficulty here is that the anisotropy measure (σ/θ) of the metric (7) is constant which means it remains anisotropic for all times. It would be interesting to find a singularity-free solution with σ/θ decreasing with t so that it can isotropise at late times to go over to FRW model. The uniqueness of the metric (8) has been established for a general orthogonal metric, separable in space and time in comoving coordinates [7]. That means to get a singularity-free model with $\sigma/\theta \neq$ constant, we will have to give up orthogonality or separability. Then the problem becomes almost unmanageable. It may however be worth its while to invest some effort on this very difficult problem.

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Appendix

For the metric (1) we introduce the tetrad $\theta^1 = A\mathrm{d}r, \theta^2 = B\mathrm{d}z, \theta^3 = C\mathrm{d}\phi, \theta^4 = A\mathrm{d}t$. The non-vanishing tetrad components R_{ab} of the Ricci tensor for (1) are given by

$$R_{14} = \frac{1}{A^2}\left[\frac{\dot{B}'}{B} + \frac{\dot{C}'}{C} - \frac{\dot{A}}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) - \frac{A'}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right]$$

$$R_{44} = \frac{1}{A^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}}{A}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{A''}{A} + \frac{A'}{A}\left(\frac{A'}{A} - \frac{B'}{B} - \frac{C'}{C}\right)\right],$$

$$R_{11} = \frac{1}{A^2}\left[\frac{A''}{A} + \frac{B''}{B} + \frac{C''}{C} - \frac{A'}{A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{C'}{C}\right) - \frac{\ddot{A}}{A} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right)\right],$$

$$R_{22} = \frac{1}{A^2}\left[\frac{B''}{B} + \frac{B'C'}{BC} - \frac{\ddot{B}}{B} - \frac{\dot{B}\dot{C}}{BC}\right]$$

$$R_{33} = \frac{1}{A^2}\left[\frac{C''}{C} + \frac{B'C'}{BC} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC}\right]$$

A prime and a dot indicate differentiation with respect to r and t respectively.

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