PRAMANA
— journal of
physics

Vol. 44, No. 4,
April 1995
pp. 295–302

Invariants of chaotic Hamiltonian systems

B R SITARAM
Physical Research Laboratory, Navrangpura, Ahmedabad 380 009, India

MS received 19 April 1994; revised 6 December 1994

Abstract. The invariants of chaotic bounded Hamiltonian systems and their relation to the solutions of the first variational equations of the equations of motion are studied. We show that these invariants are characterized by the fact that they either lose the property of differentiability as functions on phase space or that a certain formal power series defined in terms of the derivatives of the invariants has zero radius of convergence. For a specific example, we show that the former possibility appears to apply.

Keywords. Integrability; Liapunov exponents; chaos; Hamiltonian systems.

PACS Nos 03·20; 02·40

The problem of determining invariants for a Hamiltonian system has an old history. The motivation for looking for such invariants is that they help in reducing the number of degrees of freedom and thereby enable us to solve the equations of motion in terms of quadrature. Recent studies [1–4], have attacked the problem of invariants which are complex analytic functions on phase space. It is not very clear, however, why complex analyticity is required; it is thus worthwhile to see whether any light can be thrown on the structure of invariants, assuming only that they are once differentiable as functions on phase space. (The once differentiability property is of course required as invariants have to satisfy the first order PDE $\{I, H\} = 0$). We show here that the information obtained from Liapunov exponents can be used to draw certain conclusions regarding the structure of invariants in chaotic systems.

Let $\xi_i, i = 1\ldots 2n$ denote canonical coordinates on R^{2n} and let H be a Hamiltonian function such that (i) all the orbits of H are compact and (ii) for at least one orbit, one or more of the Liapunov exponents [5] are positive. We study here the problem of determining invariants for such systems and what we can learn about the properties of such invariants by looking at the Liapunov exponents.

Considered as a set of differential equations (without taking into account the special features of symplecticity), an invariant for Hamilton's equations is any solution of the equation

$$\frac{dF}{dt} = 0. \tag{1}$$

For a system with n degrees of freedom, to completely solve the problem of integrating the equations of motion, we need $2n$ functionally independent solutions of (1); of these invariants, at least one must necessarily depend on t. If we take into account the important fact that Hamilton's equations are symplectic, then, according to Liouville's theorem [6] we need to only determine n mutually involutive invariants.

(Usually, Liouville's theorem is stated in the form that n *time independent* mutually involutive invariants are required, which therefore satisfy the equation

$$\{F,H\}=0. \tag{2}$$

However, by extending phase space to include t and a conjugate variable T, and redefining H by $H+T$, we get an equivalent dynamical system and Liouville's theorem in the form usually stated applies.) In such a case, Liouville's theorem allows us to construct the remaining n invariants in a step by step manner. The converse of Liouville's theorem is also true in R^{2n}: given $2n$ independent solutions of (1), Darboux's theorem [7] allows us to choose a subset whose elements are mutually involutive.

It is easy to see that the existence of an invariant implies the existence of an exact solution of the first variational equations (the Liapunov equations). To see this, consider Hamilton's equations,

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t}=\omega_{ij}\frac{\partial H}{\partial \xi_j}\equiv\omega_{ij}H_j, \tag{3}$$

where ω is the Poisson bracket matrix. The first variational equations, whose solutions provide the Liapunov exponents, are given by,

$$\frac{\mathrm{d}\eta_i}{\mathrm{d}t}=\omega_{ij}H_{jk}\eta_k. \tag{4}$$

While solving these equations, we choose a specific solution $\xi_i(t)$ of Hamilton's equations, insert it into the rhs of (4) and determine $\eta_i(t)$; obviously, the solution so obtained will depend on the choice of orbit. Alternatively, we can look at (4) as the equations along the characteristic of the partial differential equations,

$$\frac{\partial\eta_i}{\partial t}+\omega_{jk}\frac{\partial\eta_i}{\partial\xi_j}H_k=\omega_{ij}H_{jk}\eta_k, \tag{5}$$

where, we consider η as a function of ξ and t. Knowing a solution of (4) for every orbit and for a suitable choice of initial conditions for η is equivalent to solving (5).

Now, let F be a solution of (1) and define

$$\eta_i=\omega_{ij}\frac{\partial F}{\partial\xi_j}. \tag{6}$$

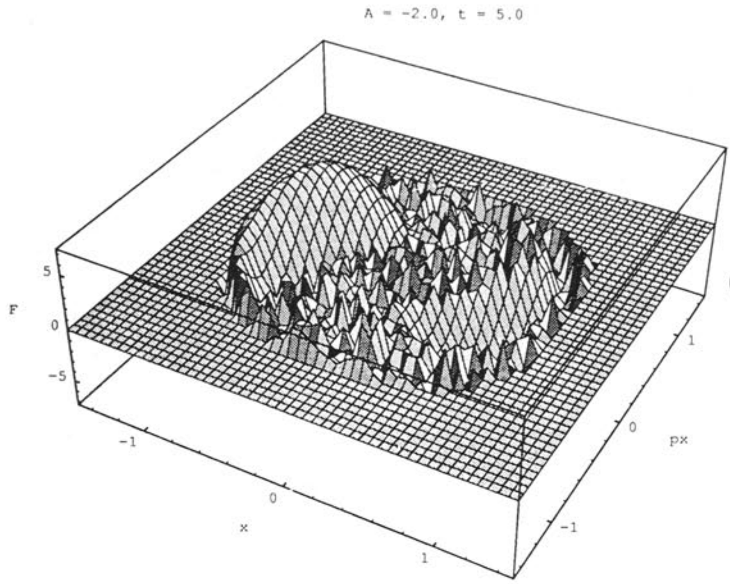
A simple substitution of this in the Liapunov equation shows that the equation is indeed valid. If F satisfies (i) F is C^∞ as a function on phase space (ii) the formal series,

$$\sum_{n=0}^{n=\infty}\varepsilon^n Ad_F^{(n)}(\xi,t), \tag{7}$$

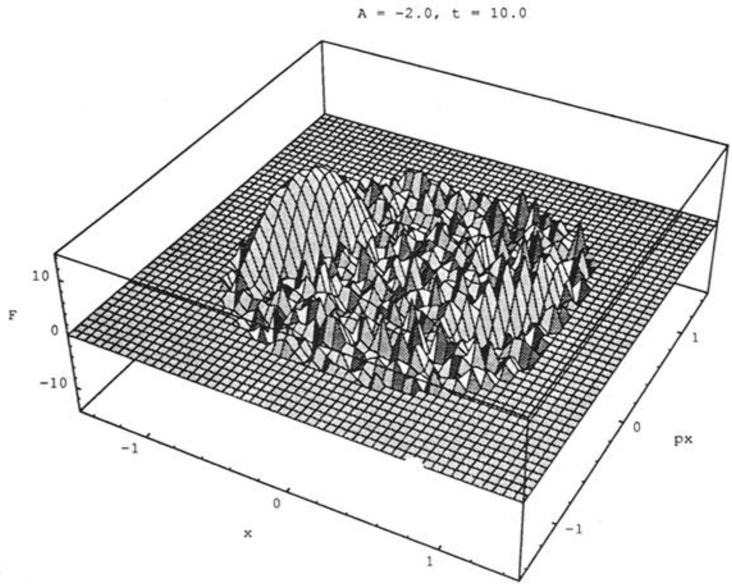
where

$$Ad_F^{(n)}(\xi,t)=\{F,Ad_F^{(n-1)}(\xi,t)\},\ Ad_F^{(1)}(\xi,t)=\{F,\xi\}, \tag{8}$$

converges in some circle in the complex ε plane around $\varepsilon=0$ for all ξ, then F defines a canonical transformation for each sufficiently small ε. This transformation has the property that it is compact; hence the corresponding Liapunov exponent is 0.



(a)

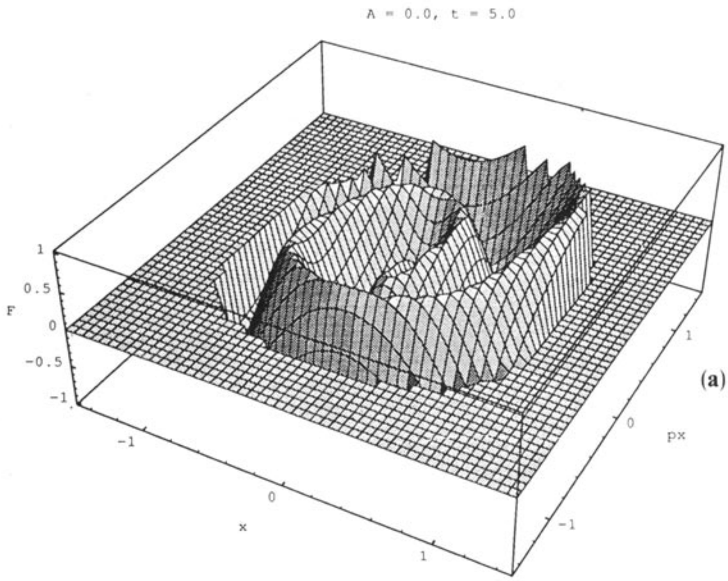


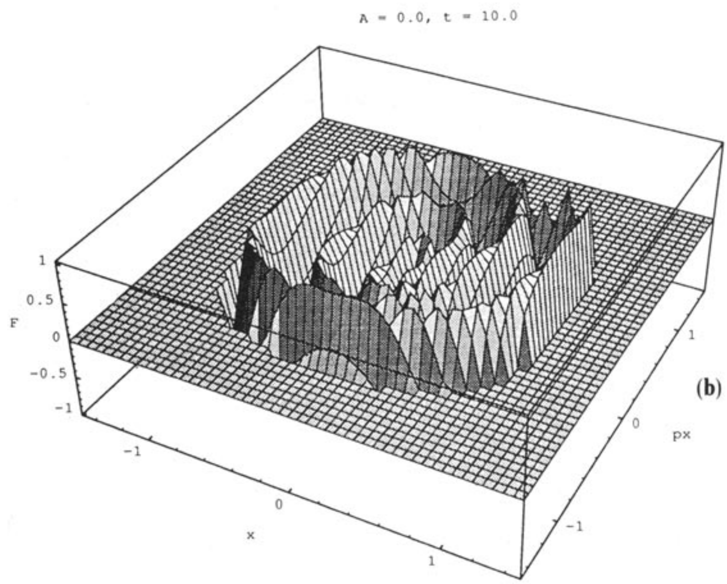
(b)

Figures 1a and 1b. Plot of F versus x and px for the Hamiltonian given by (13), with $y = 0$ and $E = 1$, for times $t = 5$ and 10 respectively. The parameter A has been chosen as -2 corresponding to the chaotic regime.

The converse of the relationship between invariants and solutions of the Liapunov equations is also true: Given a solution $\eta_i(\xi, t)$, we can construct an invariant function $F(\xi, t)$ by

$$\frac{\partial F}{\partial \xi_i} = \Omega_{ij}\eta_j, \quad \frac{\partial F}{\partial t} = H_j \eta_j, \tag{9}$$



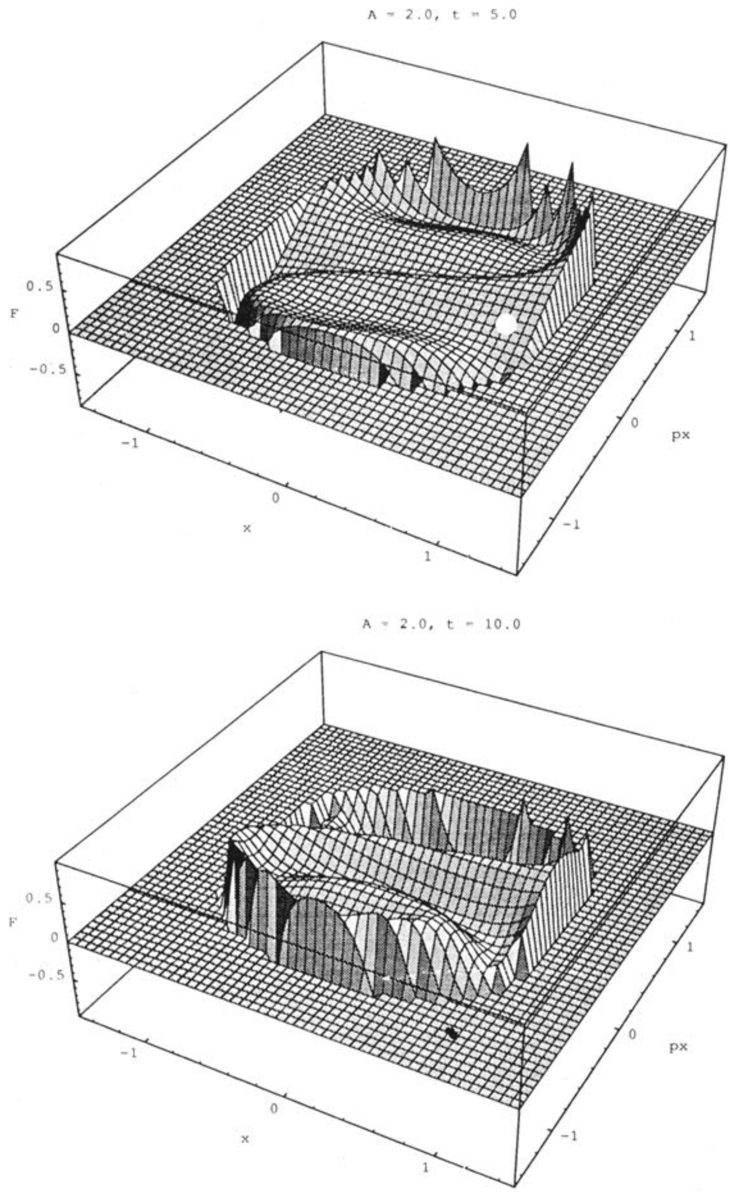


Figures 2a and 2b. Same as figures 1a and 1b, with $A = 0$ (integrable case).

where Ω is the inverse matrix of ω. By construction, F is invariant, i.e., it satisfies (1). This construction is possible, provided the matrix C_{ab} defined by

$$C_{ab} = \Omega_{aj}\frac{\partial \eta_j}{\partial \xi_b} - \Omega_{bj}\frac{\partial \eta_j}{\partial \xi_a} \tag{10}$$

vanishes. This condition is satisfied, provided C_{ab}, which is a function of ξ and t



(a)

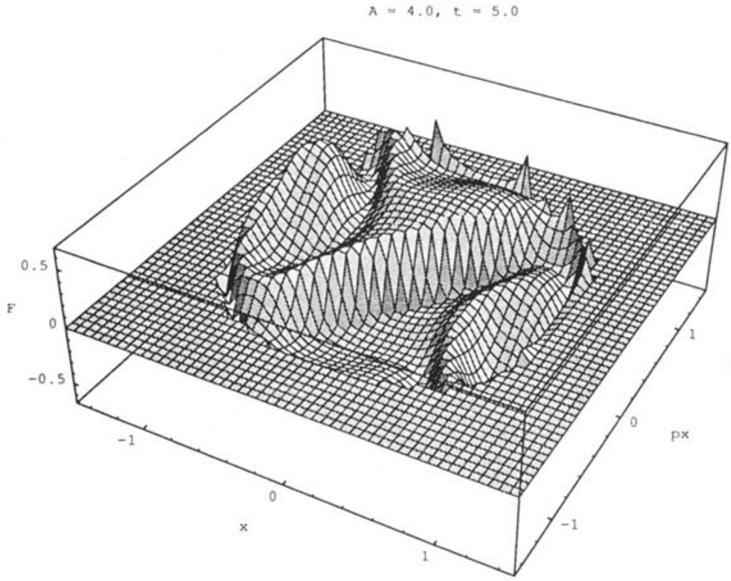
(b)

Figures 3a and 3b. Same as figures 1a and 1b, with $A = 2$ (integrable case).

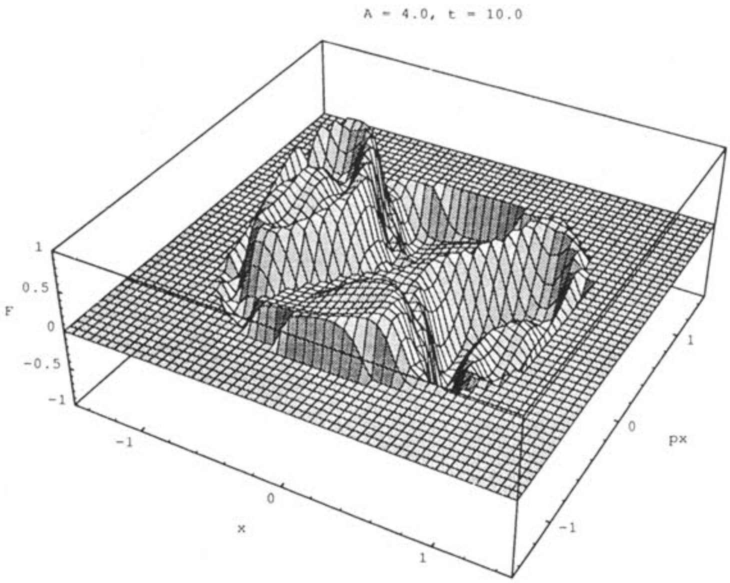
vanishes for $t = 0$, as the evolution equation for C is given by

$$\frac{\mathrm{d}C_{ab}}{\mathrm{d}t} = C_{aj}H_{jb} - C_{bj}H_{ja}. \tag{11}$$

We thus conclude that every solution of the Liapunov equations leads to an invariant, provided the initial conditions are chosen suitably. This can always be



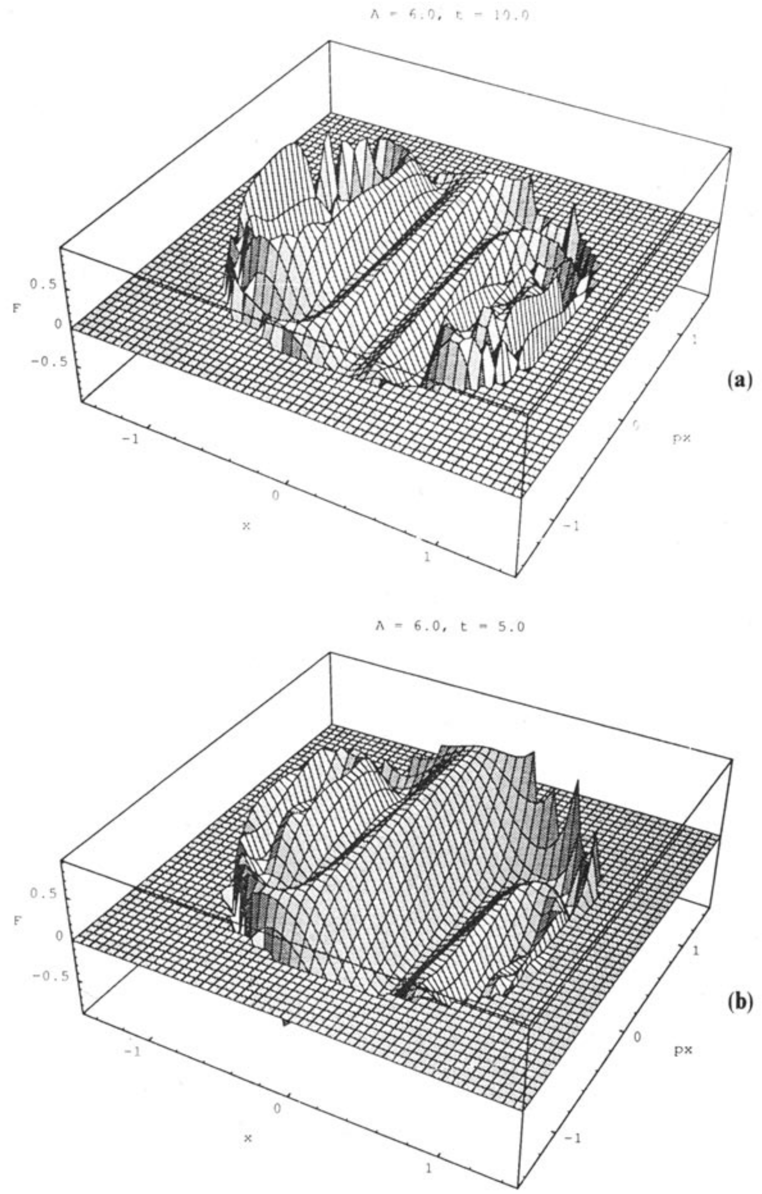
(a)



(b)

Figures 4a and 4b. Same as figures 1a and 1b, with $A = 4$ (nonintegrable case).

done; e.g., if the ξ are cartesian coordinates on phase space, a convenient choice is $\eta_i(\xi, t=0) = \delta_{ij}$ for fixed j, which corresponds to the initial condition $F(\xi, t=0) = \Omega_{ik}\delta_{kj}$. Since the Liapunov equations are linear, if the initial conditions are chosen to be linearly independent, so are the solutions for all time. But, *linear* independence of the solutions of the Liapunov equations in fact corresponds to *functional* independence of the invariants. Thus, this completes the construction of the F's,



Figures 5a and 5b. Same as figures 1a and 1b, with $A = 6$ (integrable case).

as, choosing $2n$ different initial conditions allows us to construct $2n$ functionally independent F's.

From the assumptions made regarding the non-vanishing of at least one of the Liapunov exponents, it is clear that at least one of the invariants so constructed must violate one of the two conditions given above, i.e., the invariant must be non-C^{∞} as a function on phase space or the formal power series must have zero radius of convergence. (The second condition can be re-expressed in the form that the solutions of the equations:

$$\frac{d\xi(\varepsilon)}{d\varepsilon} = \{F, \xi(\varepsilon)\}, \tag{12}$$

are singular at $\varepsilon = 0$.)

We have studied the above problem numerically for the quartic oscillator in 2 dimensions, defined by the Hamiltonian [8]

$$H = px^2 + py^2 + x^4 + y^4 + Ax^2y^2. \tag{13}$$

As is well known, the system is integrable for $A = 0, 2, 6$ and becomes increasingly chaotic as $A \to -2$. For this system, (1) was directly integrated using the initial conditions $F(x, y, px, py, t = 0) = x$, corresponding to the initial conditions $\eta_i(t = 0) = \delta_{i1}$. This choice of initial conditions can be motivated by the fact that if there is a non-zero Liapunov exponent, then we would expect almost all solutions of the Liapunov equations to be dominated by the particular solution corresponding to the positive Liapunov exponent. The values of the invariant function have been plotted on the Poincare surface defined by x, px and $y = 0$ for various times t and parameters A for the energy $E = 1$. (The Hamiltonian under consideration exhibits scaling, so that it is adequate to study one energy). As can be seen, for values of A corresponding to chaotic regimes, the values show increasing jaggedness, indicative of the fact that the phase space derivatives of F become larger and larger. We interpret this result as showing that the function eventually ceases to be analytic on phase space.

Acknowledgements

We would like to thank the referee for some useful comments.

References

[1] S L Ziglin, *Funct. Anal. Appl.* **16**, 181 (1983a)
[2] S L Ziglin, *Funct. Anal. Appl.* **17**, 6 (1983b)
[3] H Ito, *Kodai Math. J.* **8**, 120 (1985)
[4] H Yosida, *Physica* **D29**, 128 (1987)
[5] G Benettin, L Gaglani and J M Strelcyn, *Phys. Rev.* **A14**, 2338 (1976)
[6] V I Arnol'd and A Avez, *Ergodic problems of classical mechanics*, (Benjamin, New York, 1968)
[7] Y Choquet-Bruhat, C de Witt-Morette and M Dillard-Bleick, *Analysis, manifolds and physics* (North Holland, Amsterdam, 1977)
[8] M Lakshmanan and R Sahadevan, *Phys. Rep.* **224**, 1 (1993)