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Conversion of a chaotic attractor into a strange nonchaotic attractor in an one dimensional map and BVP oscillator

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Abstract. In this paper we investigate numerically the possibility of conversion of a chaotic attractor into a nonchaotic but strange attractor in both a discrete system (an one dimensional map) and in a continuous dynamical system – Bonhoeffer–van der Pol oscillator. In these systems we show suppression of chaotic property, namely, the sensitive dependence on initial states, by adding appropriate i) chaotic signal and ii) Gaussian white noise. The controlled orbit is found to be strange but nonchaotic with largest Lyapunov exponent negative and noninteger correlation dimension. Return map and power spectrum are also used to characterize the strange nonchaotic attractor.

Keywords. One dimensional map; Bonhoeffer-van der Pol oscillator; controlling of chaos; strange nonchaotic attractor.

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1. Introduction

In recent years there has been increasing interest in the study of controlling of chaotic motion in both theoretical model equations and experimental systems [1–8]. There are also recent attempts to use chaos profitably, by synchronizing chaotic orbits [9–11]. The existing control methods are capable of converting a chaotic motion into a regular dynamics either by stabilizing an unstable periodic orbit embedded in the chaotic attractor [2, 4–7] or by creating a new periodic orbit [1, 3, 7, 8]. One of the motivations for controlling chaotic motion is its extreme sensitive dependence on initial conditions. In the chaotic regime two nearby trajectories diverge exponentially until they become completely uncorrelated and hence future prediction becomes inaccurate. It is thus important to investigate the possibility of suppression of sensitive dependence on initial conditions and maintaining strangeness of the attractor instead of stabilizing a periodic orbit.

In this paper we wish to make a detailed study of conversion of chaotic attractor into a strange nonchaotic attractor by adding appropriate i) chaotic signal and ii) Gaussian noise. We carry out our investigation in both a discrete system [12, 13]

$$x_{n+1} = x_n \exp(A(1 - x_n)), \quad x_n > 0 \tag{1}$$

and in a continuous dynamical system, the Bonhoeffer-van der Pol (BVP) oscillator [7]

$$\dot{x} = x - x^3/3 - y + f\cos t, \tag{2a}$$

$$\dot{y} = c(x + a - by). \tag{2b}$$

In (2) a, b, c are constant parameters and f is the amplitude of the external periodic force. Recently, Carroll and Pecora [14] studied the effect of added chaotic and noise signals in a Duffing oscillator circuit in a different context. They studied the effect of added signal on flipping of the state of the system between two coexisting periodic attractors. Stochastic resonance has been observed. In our present study we use chaotic and noise signals in the context of controlling of chaos.

Firstly, we consider the one dimensional map (1). We fix the parameter A in a chaotic regime. Then a chaotic solution is added to the system. The dynamics of the system is studied by varying the coupling parameter. The key observation is that strange nonchaotic attractor replaces actual chaotic attractor of the map for coupling parameter value greater than certain critical value. Lyapunov exponent, power spectrum, correlation dimension are used to characterize strange nonchaotic attractor. The effect of Gaussian noise has also been studied by varying the amplitude of the noise for a fixed value of mean and standard deviation. Strange nonchaotic attractor occurs for a range of amplitude of the noise signal. These results are presented in § 2.

Secondly, we report our studies on the BVP equation (2) in § 3. The parameters a, b, c and f are fixed in a chaotic region. Chaotic solution generated from the logistic map $x_{n+1} = 4x_n(1 - x_n)$ is added to the right hand side of (2a). Here again, chaotic attractor is destroyed and strange nonchaotic attractor is found to occur for a range of amplitude of the chaotic signal. Finally, § 4 contains summary and conclusions.

2. Strange nonchaotic attractor in the map (1)

In this section we study the effect of chaotic solution and Gaussian white noise in the map (1). Map (1) is often used in theoretical studies of population growth with x_n as the population density at time n. First the dynamics of equation (1) is studied as a function of A. Bifurcation diagram 1a illustrate the period doubling route to chaos. The one dimensional Lyapunov exponent of the attractors of (1) is estimated using the relation

$$\lambda = \lim_{n \to \infty} (1/n) \sum_{i=1}^{n} \ln |f'(x_i)|, \tag{3a}$$

where

$$f'(x) = \exp(A(1 - x)) - Ax \exp(A(1 - x)). \tag{3b}$$

The variation of Lyapunov exponent against the parameter A is plotted in figure 1b. As expected λ is negative in the regular regime and is positive in the chaotic regime. Chaotic motion is found for $A = 3$. The Lyapunov exponent of this chaotic attractor is $\approx 0{\cdot}39$. Now, we investigate the effect of addition of chaotic signal to (1).

2.1 *Effect of chaotic signal*

The map with the addition of chaotic solution can be written as

$$x_{n+1} = x_n \exp(B(1 - x_n)) + Cy_n, \tag{4}$$

where y_n is the chaotic solution generated from the map (1). We fix B at 3. First we study the influence of chaotic solution generated with $A = 3$ from (1). Later, we show the effect of chaotic solution of (1) generated with different values of A.

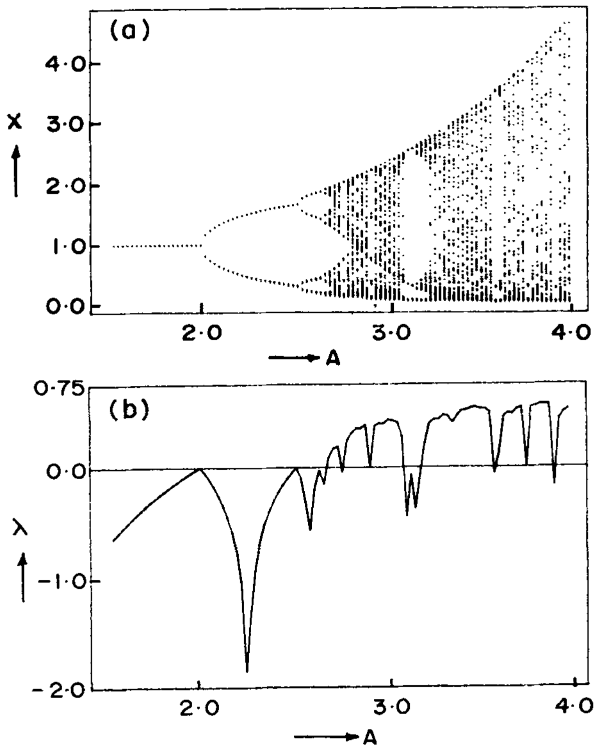

Figure 1. **a**) Bifurcation diagram showing period doubling route to chaos. **b**) Calculated Lyapunov exponent against the parameter A.

Figure 2 shows the estimated Lyapunov exponent as a function of the parameter C. From this figure we note that λ is positive for $C < C^* \approx 0{\cdot}075$ while it takes negative value for $C > C^*$. Lyapunov exponent with negative value implies that the solution is insensitive to small disturbance in the initial conditions. In other words, chaotic property is suppressed for $C > C^*$ and hence the attractor is nonchaotic. Figure 3 shows the attractor in x_n versus x_{n+1} plane for $C = 0{\cdot}1$. The Lyapunov exponent of the attractor is $-0{\cdot}12$. Further, we have estimated correlation dimension D_c of the controlled attractor. To calculate D_c we used the algorithm proposed by Grassberger and Proccacia [15]. Suppose $x_i (i = 1, 2, \ldots, N)$ represents the attractor points in the return map. We count the number of points within a ball of radius r centred at each x_i for a set of N points. The correlation function is defined by

$$C(r) = \lim_{N \to \infty} (1/N^2) \sum_i \sum_j H(r - |x_i - x_j|), \tag{5a}$$

where $H(\Theta) = 1$ for $\Theta > 0$ and $H(\Theta) = 0$ for $\Theta < 0$. Then , the correlation dimension is given by

$$D_c = \lim_{r \to 0} \ln C(r)/\ln(r). \tag{5b}$$

D_c is the slope in the $\ln C(r)$ versus $\ln(r)$ plot. The correlation dimension of the

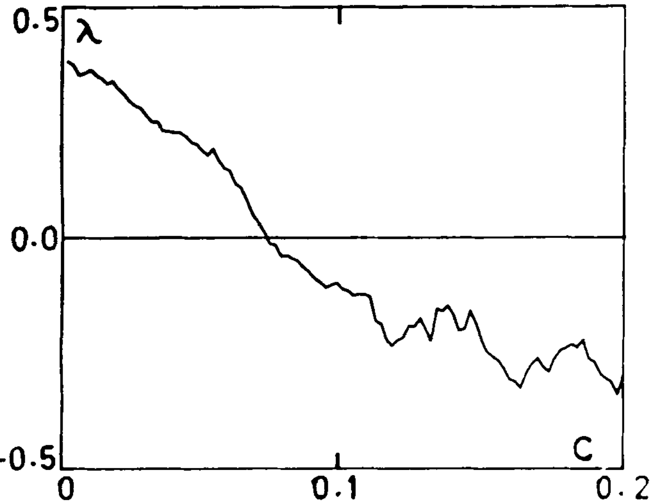

Figure 2. Estimated Lyapunov exponent versus the parameter C.

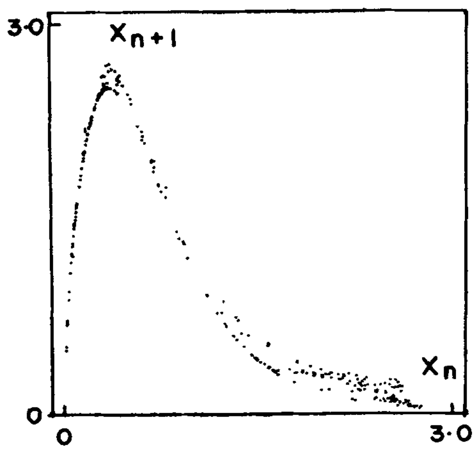

Figure 3. Strange nonchaotic attractor of equation (4) for $C = 0{\cdot}1$.

attractor shown in figure 3 is estimated as 0·82 where we have used $N = 10^4$. The attractor is thus strange. The negative Lyapunov exponent, noninteger dimension and return map (figure 3) imply that the controlled attractor is strange but nonchaotic. Strange nonchaotic attractor has been previously found to occur in numerical studies of many nonlinear systems and also in actual experiments [16–20].

The occurrence of strange nonchaotic attractor is further confirmed by power spectrum analysis. For a strange nonchaotic attractor the number of peaks $N(\sigma)$ in a power spectrum exceeding a threshold amplitude σ scale as [21]

$$N(\sigma) \propto \sigma^{-\alpha}, \quad 1 < \alpha < 2 \tag{6}$$

while for quasiperiodic attractor

$$N(\sigma) \propto \ln(1/\sigma). \tag{7}$$

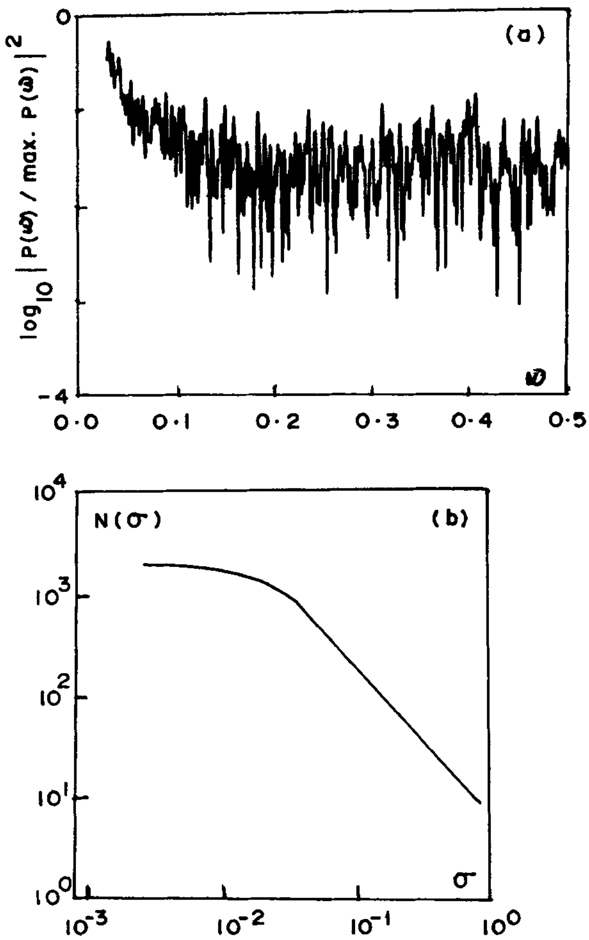

Figure 4. **a)** Power spectrum of the strange nonchaotic attractor (figure 3). **b)** The log–log plot of the threshold dependence of the number of peaks $N(\sigma)$ with the amplitude greater than the threshold σ.

Power spectrum of the attractor (figure 3) is obtained using fast Fourier transform with 4096 data points and is shown in figure 4a. The variation of $N(\sigma)$ against the threshold amplitude σ on log–log scale is plotted in figure 4b. The value of the scaling exponent α is estimated as 1·45. This is in agreement with the power-law scaling relation (6).

We have also used another measure, the invariant density which shows differences between the chaotic and strange nonchaotic attractors. We measured the densities $P(x)$ using 10^4 data points with the results shown in figure 5a for the uncontrolled chaotic orbit and figure 5b for the strange nonchaotic attractor. $P(x)$ is much more sharply defined for chaotic attractor than for the strange nonchaotic attractor. Moreover, the strange nonchaotic attractor displays a considerably smaller amplitude.

Strange nonchaotic attractor is found for C values for which $\lambda \leqslant 0$. Figure 6 shows the variation of correlation dimension as a function of the parameter C. Figures 2 and 6 clearly support the occurrence of strange nonchaotic attractor for $C > C^*$. In the map (4) the interaction between dynamic and stochastic (coupling term) forces gives

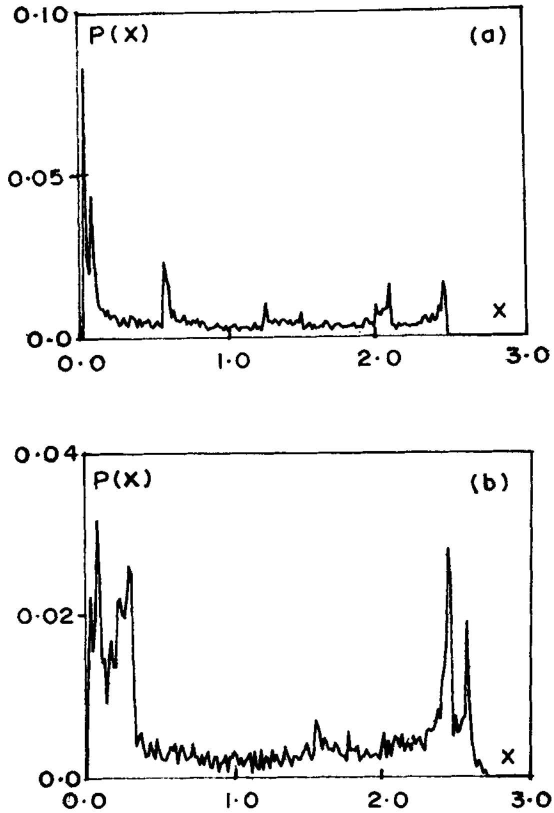

Figure 5. Calculated probability density of the **a**) uncontrolled chaotic attractor and **b**) strange nonchaotic attractor shown in figure 3.

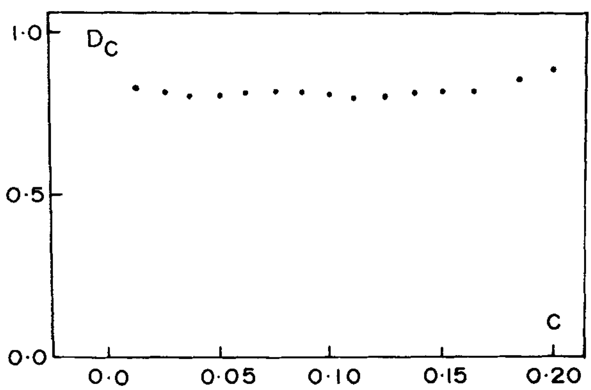

Figure 6. Variation of correlation dimension as a function of C.

rise to the strange nonchaotic attractor. The effect of various nonlinear coupling terms such as y^3, $\sin y$ and $\exp(y)$ has also been studied. The critical value C^* above which strange nonchaotic attractor occurs for the coupling terms y^3, $\sin y$ and $\exp(y)$ are found to be 0·0425, 0·168 and 0·0225 respectively. Further, we have carried out our analysis using the chaotic solution generated for various values of A. For C

values in the interval (0, 0·2) we estimated Lyapunov exponent of attractors of (4) and determined the value of C^* above which λ becomes negative. Table 1 gives the C^* value for different A values.

2.2 *Effect of Gaussian white noise*

In this subsection we study the effect of noise added to the chaotic solution of (1). The map (1) with external noise can be written as

$$x_{n+1} = x_n \exp(A(1 - x_n)) + D\eta(n), \tag{8}$$

where $\{\eta(n)\}$ are independent random numbers with mean m and standard deviation σ. We fix A at 3. We study the influence of noise by varying the amplitude D of the noise for $m = 0{\cdot}2$ and $\sigma = 0{\cdot}1$. We used the Box-Müller algorithm to generate Gaussian random variables from uniform random variables. Figure 7 shows the mean Lyapunov exponent obtained by averaging over 500 realisations of $\eta(n)$ as a function of D. For

Table 1. Estimated critical value C^* for different A values used to generate chaotic solution. The generated chaotic solution is added to equation (4).

A	C^*
2·8	0·051
2·9	0·067
3·0	0·075
3·1	0·064
3·3	0·101
3·4	0·086
3·5	0·090
3·7	0·187
3·8	0·158

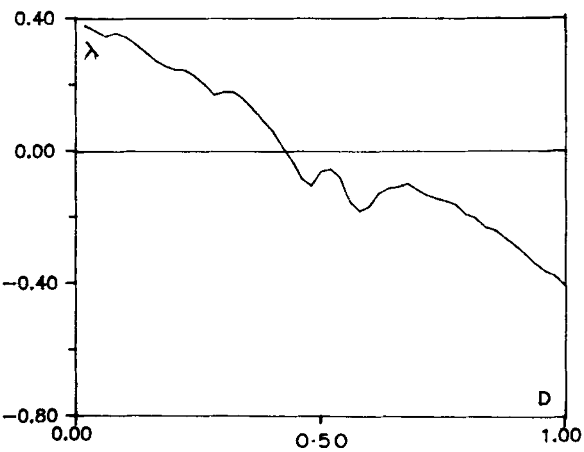

Figure 7. Estimated Lyapunov exponent versus the amplitude D.

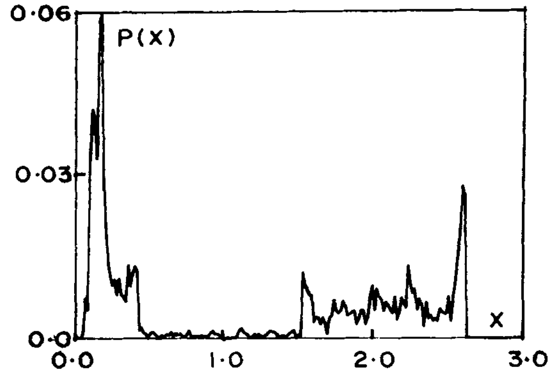

Figure 8. Calculated probability density function of the strange nonchaotic attractor with $D = 0{\cdot}6$.

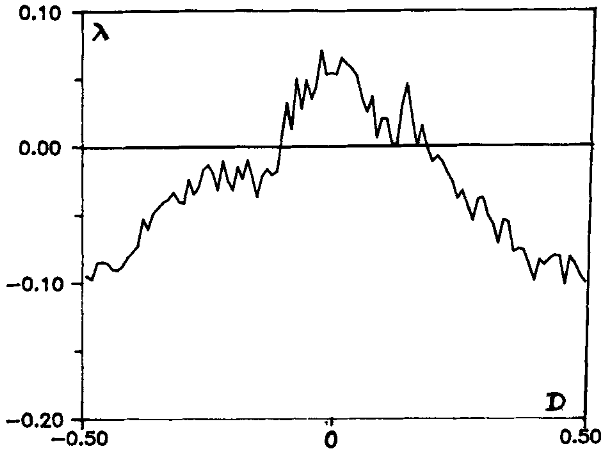

Figure 9. Maximal Lyapunov exponent as a function of D for the BVP oscillator.

$D < D^*$ the Lyapunov exponent is positive. In this case the long time motion is still chaotic. For $D > D^*$ the Lyapunov exponent is negative and the attractor is nonchaotic. We have estimated the probability density function $P(x)$ for $D = 0{\cdot}6$ and is plotted in figure 8. Here again we note that $P(x)$ is much more sharply defined for chaotic attractor (figure 5a) than for the strange nonchaotic attractor. Further, the height of the peaks are considerably smaller for strange nonchaotic attractor.

3. Strange nonchaotic attractor in BVP oscillator

The BVP oscillator with a chaotic signal added to the periodic driving force can be written as

$$\dot{x} = x - x^3/3 - y + f\cos t + Du, \tag{9a}$$

$$\dot{y} = c(x + a - by), \tag{9b}$$

where $u(t)$ is the chaotic solution generated from the logistic map

$$u_{n+1} = 4u_n(1 - u_n). \tag{10}$$

Iteration of map (10) lies in the interval (0, 1). The solution is then converted into the range $(-1, 1)$. D in (9a) is the amplitude of the chaotic solution added to the driving force. The parameters a, b, c are fixed at 0·7, 0·8 and 0·1 respectively. In the absence of $Du(t)$ period doubling phenomenon culminating in chaos is observed when f is

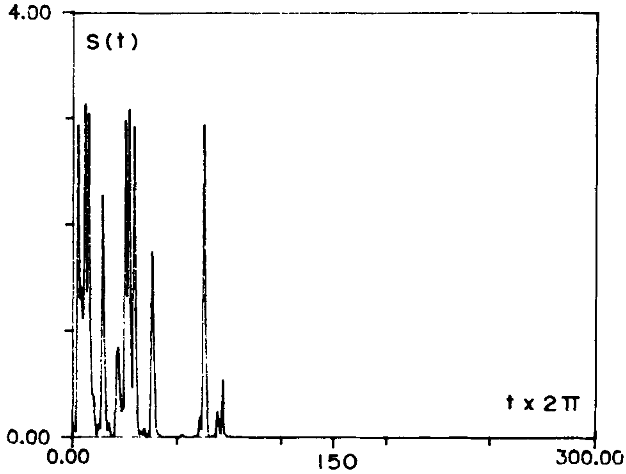

Figure 10. Calculated $S(t)$ versus t for $D = 0{\cdot}3$.

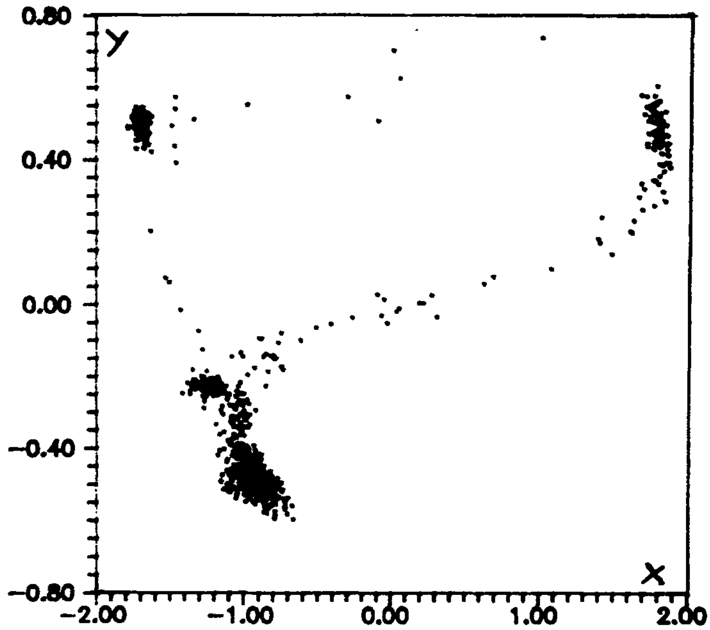

Figure 11. Strange nonchaotic attractor of the BVP equation (9) with $D = 0{\cdot}3$.

increased from small value. At $f = 0{\cdot}7182$ the system enters into a chaotic domain. We fix f at 0·74 for which a well developed chaotic motion is observed. Chaotic solution generated from the logistic map (10) is added to the system after every $(2\pi/50)$ time step. The long time behaviour of the system is analysed using Lyapunov exponent. Figure 9 shows the variation of the maximal Lyapunov exponent as a function of D. It is seen that for a range of D values λ is negative, that is, the motion is nonchaotic. We can illustrate the influence of added chaos signal more clearly by looking at the distance $S(t)$ between two orbits starting from two nearby initial conditions. The distance $S(t)$ between two trajectories starting from $X(0)$ and $X'(0)$ is given by

$$S(t) = \|X(t) - X'(t)\|.$$

For regular behaviour $S(t)$ will decay to zero in the limit $t \to \infty$. On the other hand, it will vary irregularly with time for chaotic motion. In figure 10 we report the calculated $S(t)$ for $D = 0{\cdot}3$. The $S(t)$ diminishes to zero after short-lived irregular variation. This clearly supports the loss of sensitive dependence on initial conditions. Figure 11 shows the Poincaré map of the attractor for $D = 0{\cdot}3$. The λ of the attractor is negative as seen from figure 9. From the negative value of the Lyapunov exponent and Poincaré map (figure 11) we conclude that the controlled attractor is strange nonchaotic. Strange nonchaotic attractor has been observed in the BVP equation when appropriate Gaussian white noise is added to the system instead of chaos [7].

4. Summary and conclusions

In this paper we have studied the effect of external chaotic signal and noise term in an one-dimensional map and BVP equation. To these systems chaos from deterministic dynamical systems is added. However, to the map (1) chaotic solution from the same system is added whereas chaotic signal from a different dynamical system is added to the BVP equation. When a chaotic solution is added to these systems strange nonchaotic attractor is found to replace the strange chaotic attractor for a range of coupling strength. Strange nonchaotic attractor is also observed when Gaussian noise is added instead of chaotic signal. Though the controlled orbit still appears complex, it is structurally stable and small errors in an initial conditions will not have strong effect on the long time prediction. Further, chaos is not always an unwanted phenomenon. It can be utilized for useful purposes. In such a case, conversion of a chaotic attractor to a strange nonchaotic attractor may provide much better predictability.

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