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String theory in five dimensional cosmological models

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Abstract. A study on string theory has been done in five dimensional flat space-time. Barotropic equation of state and p-string model are discussed. Also a polynomial relation between the two scale factors is assumed. In some special cases the solution reduces to generalized Kasner metric. Further diminision of extra dimension with the evolution of universe is exhibited. A detailed study of phase-space analysis is done for geometric string model.

Keywords. String theory; higher dimension; contraction; phase space.

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1. Introduction

The unification of gravitational forces with other forces in nature is not possible in the usual 4-dimensional space-time. So higher dimensional theory may be useful to meet this challenge in quantum field theory. Recently, the development of super string theory [1] and 10 dimensional $N = 1$ Yang-Mills supergravity in its field theory limit suggest the importance of theories in more than 4-dimensional space-time. Also the experimental detection of time variation of fundamental constants provide strong evidence for the existence of extra-dimensions [2]. Moreover, in recent years the variable gravity 5D theory of Wesson [3] deserves serious attention. Here the 4D space-time is extended to 5D space-time-mass in which the fifth co-ordinate is the rest mass. If the rest mass is treated as constant then the theory reduces to general relativity and is thus an extension of Einstein's theory of gravitation.

In cosmology, this higher dimensional theory might be useful at the early stages of the evolution of the universe. It is generally assumed that the extra dimensions contract to the Planckian length scale or remain constant with the evolution of the universe. Higher dimensional solutions both for cosmological and non-cosmological cases have been performed recently by several workers [4–8]. String theory does not seem to have been considered in higher dimension earlier. As string concept is useful before the particle creation and can explain galaxy formation, it is interesting to study string cosmology in higher dimensional theory.

In this addendum we have studied string theory for flat Kaluza–Klein metric. A polynomial relation is assumed between the metric coefficients to solve the field equations. The barotropic equation of state and p-string model are considered for separate solutions. Some solutions correspond to generalized Kasner metric. Further the field equations reduce to an autonomous system for geometric string model and phase-space solutions are studied in detail.

2. Einstein field equations and their solutions

The five dimensional line element due to Gron [8] is

$$dS^2 = -dt^2 + R^2(dx^2 + dy^2 + dz^2) + A^2 dm^2, \tag{1}$$

Unlike Wesson [3], the fifth co-ordinate is taken to be space-like and the metric co-efficients are assumed to be functions of time only.

The energy momentum tensor for the string-dust system is [9],

$$T_{\mu\gamma} = \rho V_\mu V_\gamma - \lambda x_\mu x_\gamma, \tag{2}$$

where ρ is the rest energy density of the cloud of strings with particles attached to them and λ is the tension density of the string cloud. The five-velocity vector V^μ has the components $V^\mu = (1,0,0,0,0)$ and x^μ represents string's direction which is taken along the extra dimension i.e. $x^\mu = (0,0,0,0,A^{-1})$. Also V^μ and x_μ will satisfy

$$V^\mu V_\mu = -x^\mu x_\mu = -1 \text{ and } V^\mu x_\mu = 0.$$

So the non-vanishing components of the Einstein's equation

$$R_{\mu\gamma} - \tfrac{1}{2} R g_{\mu\gamma} = -T_{\mu\gamma},$$

are

$$3\frac{\dot{R}^2}{R^2} + 3\frac{\dot{R}}{R}\frac{\dot{A}}{A} = \rho, \tag{3}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 2\frac{\dot{R}\dot{A}}{RA} + \frac{\ddot{A}}{A} = 0, \tag{4}$$

and

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = \lambda. \tag{5}$$

The Bianchi identity is

$$\dot{\rho} + 3\frac{\dot{R}}{R}\rho + (\rho - \lambda)\frac{\dot{A}}{A} = 0. \tag{6}$$

We shall now discuss solutions of these field equations for two different equation of state for string model.

Case I: Barotropic equation of state

Here ρ and λ are connected by an arbitrary relation

$$\rho = \rho(\lambda).$$

Let us assume

$$A = \mu R^n, \quad \mu, n \text{ are constants}, \tag{7}$$

between the scale factors for unique solution of the field equations and we obtain solutions as follows

$$
\begin{aligned}
R &= R_0(t - t_0)^{\alpha}, \\
A &= A_0(t - t_0)^{n\alpha}, \\
\rho &= 3(n + 1)\alpha^2(t - t_0)^{-2}, \\
\lambda &= \frac{(1 - n^2)\alpha}{(3 + 2n + n^2)}(t - t_0)^{-2}.
\end{aligned}
\tag{8}
$$

Here t_0 and R_0 are arbitrary constants and

$$
\begin{aligned}
A_0 &= \mu R_0^n, \\
\alpha &= (2 + n)/(3 + 2n + n^2).
\end{aligned}
$$

We note that if $n > 0$ then R and A behave identically. $t = t_0$ is the initial epoch and it is a point singularity. With the evolution of the universe the two scale factors increase with time while string and particle energy density gradually decrease (for $n > 1$ string density is unphysical).

For $n = 0$ the scale factor corresponding to extra-dimension is constant in time, otherwise it is identical to $n > 0$.

If $n < 0$, the initial epoch $t = t_0$ is a line singularity and with the evolution of the universe R increases while A decreases. Thus extra-dimension becomes insignificant as the time proceeds after the creation and we are left with real four dimensional world. For $n = -1$, $\rho = \lambda = 0$ and the solution then corresponds to generalized Kasner metric [10]. Further, for $n < -2$ the solution corresponds to contracting model of the universe, which is not of much interest here.

Case 2. Takabayashi string

Here the equation of state is [11]

$$\rho = (1 + W)\lambda, \tag{9}$$

with $W > 0$, a constant.

Now eliminating ρ and λ from (3) and (5) using (9) we obtain

$$A = CR^{(W-2)/3}\dot{R}^{(1+W)/3}. \tag{10}$$

If we substitute A from (10) in the field equations (4), we obtain a differential equation in R which is complicated in nature and difficult to solve. So for simplicity we take $W = 2$. Then from (10)

$$A = C\dot{R}, \tag{11}$$

and the corresponding differential equation in R is

$$\frac{\dddot{R}}{\dot{R}} + 4\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = 0. \tag{12}$$

For particular choice of the integration constant we have the following two sets of solutions

$$R \propto (t-t_0)^{1/2}, \quad \rho \propto (t-t_0)^{-2}$$
$$A \propto (t-t_0)^{-1/2}, \quad \lambda \propto (t-t_0)^{-2}, \tag{13}$$

and

$$R \propto (t-t_0)^{2/3}, \quad \rho \propto (t-t_0)^{-2},$$
$$A \propto (t-t_0)^{-1/3}, \quad \lambda \propto (t-t_0)^{-2}. \tag{14}$$

The first set of solutions is the well-known generalized Kasner metric [8]. As before $t = t_0$ is the initial instant of the universe in both the cases and with the expansion of the universe the extradimension and the string concept gradually dies out. So we are left with the real four dimensional universe.

3. Autonomous system

We shall now consider the autonomous system for the flat case $k = 0$. In geometric string model the field equations (3) – (5) become

$$3\frac{\dot{R}^2}{R^2} + 3\frac{\dot{R}}{R}\frac{\dot{A}}{A} = \rho,$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 2\frac{\dot{R}}{R}\frac{\dot{A}}{A} + \frac{\ddot{A}}{A} = 0,$$

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = \rho.$$

These equations can be interpreted as the field equations for five dimensional space-time with string membrane along the extra dimension [12]. The change of variables

$$x = \dot{R}/R, \quad y = \dot{A}/A,$$

reduce the about field equations (after eliminating ρ) in the canonical form

$$\dot{x} = -x^2 + xy,$$
$$\dot{y} = -x^2 - y^2 - 4xy. \tag{15}$$

It has only one critical point (0,0) which is degenerate in nature. To analyse this dynamical system we introduce

$$u = x, \quad v = y/x, \quad \frac{d\tau}{dt} = |x| = \varepsilon x \quad (\varepsilon = \pm 1). \tag{16}$$

Then from (15) and (16)

$$\frac{du}{d\tau} = \varepsilon u(-1+v),$$

$$\frac{dv}{d\tau} = -\varepsilon(2v+1)(v+1). \tag{17}$$

The particular solutions of this system are

$$u = u_0 \exp[(v_i - 1)], \quad v_i = -1 \text{ or } -\tfrac{1}{2}. \tag{18}$$

These solutions correspond to the special solutions of the system (15) and can be interpreted as straight lines in the xy-plane. Also these solutions approach the degenerate critical point $(0,0)$ along the tangent vectors $(1, v_i)$.

We shall now discuss in detail the rectilinear solutions (18) which satisfy

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \varepsilon u(v - 1).$$

In the old variables we have

$$\frac{\dot{x}}{x^2} = (v_i - 1), \quad y = v_i x,$$

whose solutions are

$$x = \frac{-1}{(v_i - 1)t - 1/x_0}, \quad y = \frac{-1}{(1 - 1/v_i)t - (1/v_i x_0)}. \tag{19}$$

Here $(x_0, y_0 = v_i \cdot x_0)$ are the initial conditions in phase space. Therefore the scale factors have the expressions

$$R = K[(v_i - 1)t - 1/x_0]^{1/(1 - v_i)},$$
$$A = K[(1 - 1/v_i)t - 1/v_i \cdot x_0]^{(v_i/1 - v_i)}. \tag{20}$$

For expanding universe, the scale factor R should be an increasing function of time while the scale factor A corresponding to extra dimension should decrease or be a constant to have compactification of extra dimensions. This is possible for both $v_i{:}v_i = -\frac{1}{2}$ or -1. Thus the phase space analysis exhibits solutions which represent expanding universe with contraction of extradimensions.

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