PRAMANA
— journal of
physics

© Printed in India

Vol. 43, No. 6,
December 1994
pp. 431–442

Controlling chaos: Stabilization of fixed points and creation of new stable attractors in 1-D maps

SANGEETA D GADRE and V S VARMA
Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India

MS received 16 March 1994; revised 2 September 1994

Abstract. In this paper we have tried to stabilize the unstable fixed points for a class of 1-D maps by using a multiplicative nonlinear feedback control mechanism. We have also used such control to create new attractors (which did not exist in the original system), to suit our requirement. The control is also found to work in the presence of noise.

Keywords. Controlling chaos; unstable fixed points; Lyapunov exponents; transient time; noise.

PACS No. 05·45

1. Introduction

Chaos in physical systems was believed to be unreliable, uncontrollable and therefore undesirable until Pecora and Carroll [1, 3] and Ott, Grebogi and Yorke [4, 5] demonstrated that chaos is manageable, exploitable and even invaluable. A chaotic system in general, cannot be made to converge to a freely evolving desired trajectory, whether periodic or chaotic, due to the inherent unpredictability of the system. The control of chaos in this context consists of forcing the system to evolve along a desired trajectory. Pecora and Carroll succeeded in achieving this by using a suitable drive variable. They demonstrated that two identical chaotic systems driven by a common signal display asymptotic convergence of their trajectories even though they may have started from very different initial conditions. They could do this provided the Lyapunov exponents of the driven system were all negative. Such behaviour was not only demonstrated numerically but also confirmed experimentally.

Ott, Grebogi and Yorke (OGY) succeeded in forcing a chaotic system on to one of its own unstable periodic orbits by making a set of small time-dependent perturbations on the system parameters in such a way that the desired periodic orbit was stabilized. They demonstrated their method numerically by controlling the Hénon map. They also pointed out that as an infinite number of unstable periodic orbits are embedded in a chaotic attractor, the stabilization of these unstable periodic orbits leads to an enhancement of the system's performance as well as to the adaptability of the system to varying performance requirements. The effectiveness of the OGY method of control has also been confirmed experimentally in several systems [6–8].

Güémez and Matías [9] suggested a new method for controlling chaos in the case of iterated maps. They stabilized a given unstable periodic orbit by applying series of regular proportional feedbacks in the form of pulses to the variable of the map. This method does not change system parameters. They illustrated it with an application to the logistic and exponential maps.

In this paper we have tried to stabilize the unstable fixed points of a map by modifying the equation which controls the dynamics of the system. Using such control we have also tried to create new attractors which did not already exist in the system. This leads to the stabilization of the system to arbitrary trajectories and not just to already existing unstable trajectories. We have also tested our control in the presence of noise and found it to be working effectively. We have restricted ourselves to those 1-D maps which are of the form $F(\lambda, x) = \lambda G(x)$ or can be put in this form by a suitable transformation [10–12]. We have chosen the logistic map as an example and studied the behaviour of this map modified by the chosen control mechanism.

2. Exponential control

Consider a discrete deterministic nonlinear dynamical system governed by

$$x_{n+1} = F(\lambda, x_n) = \lambda G(x_n). \tag{1}$$

Here x_n is the response of the system at the time n and λ is the parameter which remains constant and determines the asymptotic behaviour of a typical trajectory of the attractor. The dynamics is assumed to be known and expressed exactly through the nonlinear function F. Such systems may display periodic, quasiperiodic or chaotic behaviour depending on the value of the control parameter λ. As mentioned in the introduction, one of the aims of this paper is to stabilize the existing unstable fixed points of periodicity say k i.e. $(x_1^*, x_2^*, \ldots, x_k^*)$ of the map F. The stabilization is done by replacing the parameter λ in (1) by the multiplicative nonlinear feedback control

$$\lambda^* \exp\left(\varepsilon \prod_{i=1}^{k} (x_n - x_i^*) \right) \tag{2}$$

which becomes equal to λ^* as soon as x_n becomes equal to any of the x_i^*s. Here ε measures the stiffness of the control mechanism and λ^* is the value of the parameter corresponding to the unstable fixed points $(x_1^*, x_2^*, \ldots, x_k^*)$ of the unmodulated map F to which the system is to be made to converge. In the presence of such control, the dynamics of the modulated system is governed by

$$x_{n+1} = F\left(\lambda^* \exp\left(\varepsilon \prod_{i=1}^{k} (x_n - x_i^*) \right), x_n \right) \equiv f(\lambda^*, \varepsilon, x_n). \tag{3}$$

The form of the modulated system f is such that it is almost a trivial exercise to show that the fixed points of period k of $F(\lambda^*, x)$, are also the fixed points of the same periodicity k of the corresponding $f(\lambda^*, \varepsilon, x)$ (although f may possess other fixed points in addition). The slope α of the map $f^k(\lambda^*, \varepsilon, x_n)$ at any one of the fixed points $x_r^*, r = 1, 2, \ldots, k$, of the system is given by

$$\begin{aligned} \alpha(\varepsilon) &= \left.\frac{\mathrm{d}f^k}{\mathrm{d}x}\right|_{x_r^*} \\ &= \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_k^*} \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_{k-1}^*} \cdots \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_1^*} \end{aligned}$$

$$= \prod_{j=1}^{k} \left(\lambda^* \frac{dG}{dx}\bigg|_{x_j^*} + \lambda^* \varepsilon G(x_j^*) \prod_{\substack{i=1\\ i\neq j}}^{k} (x_j^* - x_i^*) \right)$$

$$= \prod_{j=1}^{k} \frac{dF}{dx}\bigg|_{x_j^*} + \varepsilon g(\varepsilon, \lambda^*)$$

$$= \alpha(0) + \varepsilon g(\varepsilon, \lambda^*),$$

where

$$\alpha(0) = \prod_{j=1}^{k} \frac{dF}{dx}\bigg|_{x_j^*} = \frac{dF^k}{dx}\bigg|_{x_r^*}$$

is the slope of the unmodulated map F at $x = x_r^*$ and $\varepsilon g(\varepsilon, \lambda^*)$ is a polynomial in ε of degree k. When a fixed point x_r^* of the map F with periodicity k becomes unstable through a period doubling route to chaos then $dF^k/dx|_{x_r^*} < -1$. For x_r^* to be a stable fixed point of f^k we must have $-1 < \alpha(\varepsilon) < +1$. When k is odd then $\alpha(\varepsilon) \to \infty$ as ε tends to either $+\infty$ or $-\infty$. Also $\alpha(0) < -1$. Hence there exists a range of ε for which $-1 < \alpha(\varepsilon) < +1$ and x_r^* becomes a stable fixed point of f^k for ε in this range. Similarly, when k is a multiple of 4, it can be shown that $\alpha(\varepsilon) \to +\infty$ as $\varepsilon \to \infty$. However, for other values of k, $\alpha(\varepsilon)$ may not lie in the open interval $(-1, 1)$ for any value of ε, depending on the form of the function F and values of the x_r^*s. In such cases the control will not be able to force the system to converge to the unstable fixed points $\{x_r^*\}$. Whereas for unstable fixed points x_r^* of the map F with periodicity k, created by tangent bifurcation $dF^k/dx|_{x_r^*} > 1$. When k is odd then $\alpha(\varepsilon) \to -\infty$ as ε tends to either $-\infty$ or $+\infty$. Also $\alpha(0) > 1$. Hence there exists a range of ε for which $\alpha(\varepsilon)$ lies in the range $(-1, 1)$ and x_r^* becomes a stable fixed point of f^k for ε in this range. Similarly, when k is even but not a multiple of 4, it can be shown that $\alpha(\varepsilon) \to -\infty$ as $\varepsilon \to \infty$.

Some other features of our choice of control are:

(i) the control becomes passive once the desired goal $(x_1^*, x_2^*, \ldots, x_k^*)$ is achieved. If fluctuations drive the system off the desired orbit, the control reactivates itself;
(ii) as the control is a positive unbounded function, it can take any value lying between 0 and ∞. Consequently the modulated system $f(\lambda^*, \varepsilon, x)$ may or may not remain bounded for $0 \leqslant x \leqslant 1$, and this depends on the form of the unmodulated function $F(\lambda^*, x)$. For example, for $F(\lambda^*, x) = 4\lambda^* x(1 - x)$, the modulated system remains bounded for $k = 1$ and $0 \leqslant x \leqslant 1$ in the desired range of ε, whereas for $F(\lambda^*, x) = 1 - \lambda^* (2x - 1)^2$ this is not so;
(iii) the control also destabilizes stable fixed points F for some values of ε;
(iv) the form of the control is such that the unstable fixed point $x^* = 0$ cannot be stabilized by putting $x^* = 0$ in (2). The reason is obvious. df/dx evaluated at such a point is the same as $dF/dx|_{x=0}$. So the Lyapunov exponent of the modulated map f given by eq. (3) remains the same as that of the unmodulated map F given by eq. (1). However, $x^* = 0$ may get stabilized in the process of stabilizing a non-zero unstable fixed point of the map as can be seen in the logistic map (§ 3·1).

A quantity of obvious interest in the context of controlling chaos is the time required for the system to settle on to the desired orbit. This of course depends on the stiffness of the control ε. For a given ε we study the length of the transient τ required for the system to approach within a distance ω of the desired orbit starting from some initial point. If ω_0 is the initial distance from the desired orbit, then it is clear that τ and ω are related by $\omega = \omega_0 \exp(\Lambda\tau)$, where Λ is the real part of the largest Lyapunov exponent. The slope of the plot τ against $\ln(\omega/\omega_0)$ is nothing but $1/\Lambda$. The values of the Lyapunov exponents determined in this manner are found to be in good agreement with those obtained analytically. We also study the transient time τ required for settling on to the desired orbit within a given accuracy ω as a function of ε and find that it exhibits the same behaviour as the Lyapunov exponent as a function of ε for that orbit.

3. Behaviour of the logistic map while stabilizing an unstable fixed point of period 1

The logistic map with control (2) is given by

$$x_{n+1} = 4\lambda^* \exp(\varepsilon(x_n - x_u^*))x_n(1 - x_n) \equiv f(\lambda^*, \varepsilon, x_n) \tag{4}$$

where $0 < x_n < 1$, $0 < \lambda^* < 1$ and $x_u^* = 1 - 1/(4\lambda^*)$ is an unstable fixed point of period one for $0{\cdot}75 < \lambda^* \leqslant 1$. The variation of $\mathrm{d}f/\mathrm{d}x|_{x_u^*}$ with λ^*, for different values of ε, is shown by circled lines in figure 1. The figure shows that small values of ε can only stabilize small values of x_u^* while larger x_u^*s can be stabilized by increasing ε. Although a large stiffness constant stabilizes large x_u^*s it also destabilizes small x_u^*s. Furthermore, any x_u^* can only be stabilized provided ε lies within a range $\varepsilon_{\min} < \varepsilon < \varepsilon_{\max}$. The end points of this range, i.e. $\varepsilon_{\min}$ and $\varepsilon_{\max}$, can be obtained by using the conditions $\mathrm{d}f/\mathrm{d}x|_{x_u^*} > -1$ and $\mathrm{d}f/\mathrm{d}x_{x_u^*} < 1$. For the logistic map the range of values of ε, for which x_u^* becomes a stable fixed point as a result of the modulation by the control

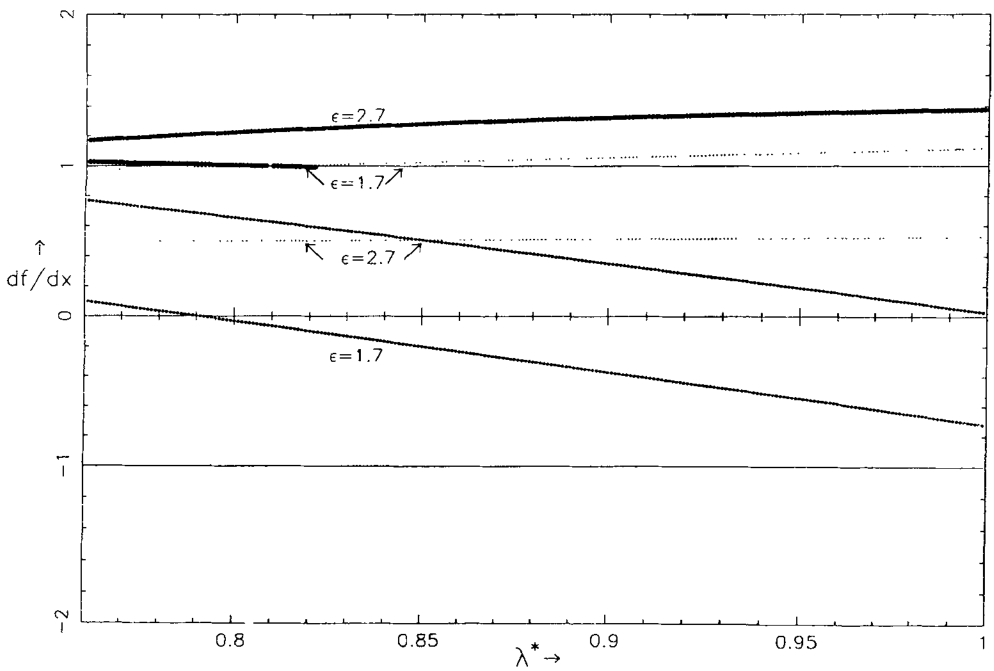

Figure 1. The variation of $\mathrm{d}f/\mathrm{d}x$ for the logistic map evaluated at $x = 0$, $\tilde{x}$ and x_u^* and λ^* is represented by dotted, bold and circled lines respectively for $\varepsilon = 1{\cdot}7$ and $2{\cdot}7$.

(2) is $(-3+4\lambda^*)/(-1+4\lambda^*)<\varepsilon<4\lambda^*$. This range of ε for which x_u^* becomes a stable fixed point of the modulated map using the control (2) is plotted against λ^* in figure 2.

The form of $f(\lambda^*,\varepsilon,x)$ for the logistic map is such that the two fixed points of F i.e. $x=0$ and x_u^* are also the fixed points of f for $0\leqslant x\leqslant 1$. For a given ε, there exists a critical value λ_c of λ^* for which $\mathrm{d}f/\mathrm{d}x|_{x=0}$ is exactly equal to 1. For $\lambda^*<\lambda_c$, $\mathrm{d}f/\mathrm{d}x|_{x=0}<1$, hence the function $f(\lambda^*,\varepsilon,x)$ lies below the line $y=x$ for small x in the x,f plot. Therefore $f(\lambda^*,\varepsilon,x)$ must intersect the line $y=x$ at least at one point $\tilde{x}$ between $x=0$ and $x=x_u^*$ (which is not true of the unmodulated map). Thus for a given ε, for $\lambda^*>\lambda_c$ the map f has only two fixed points $x=0$ and x_u^*, whereas for $\lambda^*<\lambda_c$, f possesses a third fixed point $\tilde{x}$. The plots of $\mathrm{d}f/\mathrm{d}x$, evaluated at $x=0,\tilde{x}$ and x_u^*, against λ^* for different ε have been shown by dotted, bold and circled lines respectively in figure 1. Note that for $\varepsilon=1{\cdot}7$, $\lambda_c\simeq 0{\cdot}82$ and $\mathrm{d}f/\mathrm{d}x|_{\tilde{x}}$ (bold line) stops there because $\tilde{x}$, as explained above, ceases to exist for values of $\lambda^*>0{\cdot}82$.

An important feature of our control is that although, in general, such modulation can make the system unbounded (even though the unmodulated system is bounded), this is not the case for the logistic map for $\varepsilon_{\min}<\varepsilon<\varepsilon_{\max}$. This can be shown by calculating $f(\lambda^*,\varepsilon,x_{\max})$ at different values of λ^* and ε, where $x_{\max}=1/2-1/\varepsilon+\sqrt{(1/4+1/(\varepsilon^2))}$ is the value of x at which f has an extremum. Thus, within the specified range of ε, all itineraries of the logistic map in the presence of our control mechanism always remain bounded in [0, 1]. Consequently the basin of attraction of x_u^*, which is the only stable fixed point of f for $\lambda^*>\lambda_c$, is the entire domain of x for such values of λ^*. For $\lambda^*<\lambda_c$, there are two stable fixed points $x=0$ and $x=x_u^*$. The domains

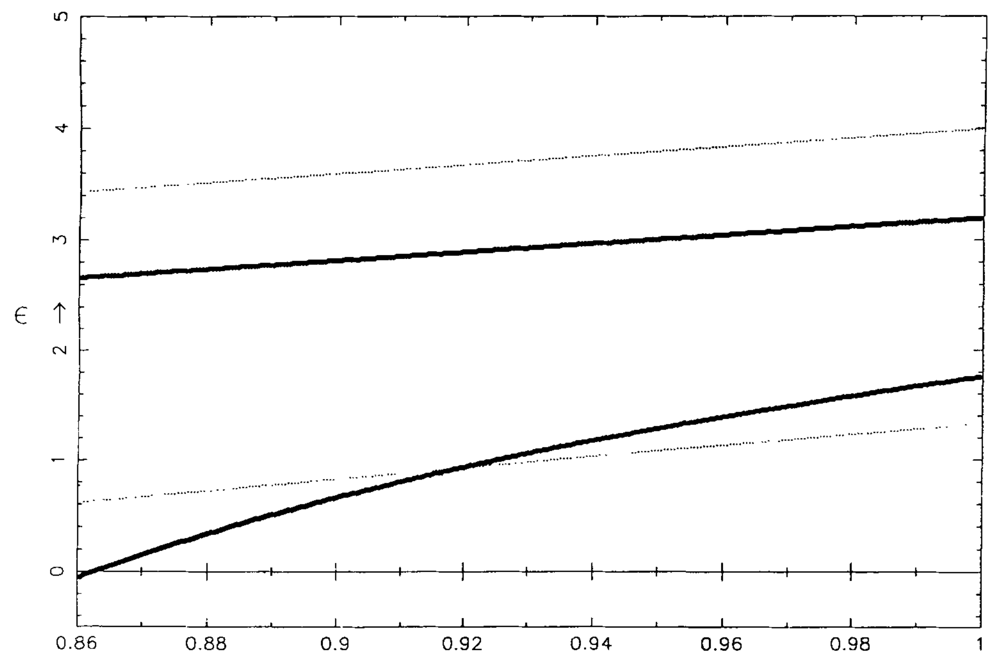

Figure 2. The range of ε for which unstable fixed points of the logistic map with periodicities 1 and 2 can be stabilized for different λ^*, using the exponential control, is represented by dotted and bold lines respectively.

of attraction of the two stable fixed points are disjoint and between them exhaust the entire domain of x. It is found numerically that the basin of attraction of $x = 0$ is $0 < x < \tilde{x}$ and $x_t < x < 1$, where x_t is the value of x such that $f(x_t) = f(\tilde{x}) = \tilde{x}$ and the basin of attraction of x_u^* is $\tilde{x} < x < x_t$. These are shown in figure 3.

Another feature worth noticing is that f remains bounded in [0, 1] even for some values of $\lambda > 1$ for which F, i.e. the logistic map, is unbounded. Therefore the unstable period-one fixed point x_u^* of the logistic map can also be stabilized for $\lambda^* > 1$. The analysis is the same as for $\lambda^* < 1$. Thus, with the present control, it is possible to make the system bounded even for those values of λ for which the unmodulated system is unbounded, and the system can be made to converge to the corresponding fixed points if it is so desired.

For a given λ^* and ε, we study the transient time τ required by the system for settling on to the fixed point x_u^* of the map. We plot τ as a function of $\ln(\omega/\omega_0)$ in figure 4 for $\lambda = 0{\cdot}9$ and $\varepsilon = 2{\cdot}2$. From the slope of the graph, the Lyapunov exponent $= -0{\cdot}133$ which is in good agreement with the value $-0{\cdot}13036$ obtained analytically. We also plot τ against ε for fixed λ and ω in figure 5 and find that τ is a decreasing function of ε. However, there is an optimum stiffness of control, beyond which increasing ε increases τ. This behaviour of τ is the same as that of the Lyapunov exponent for the map as a function of ε. The optimum value of ε corresponds to that value of ε for which the Lyapunov exponent is minimum. This is found to be a general feature of the control.

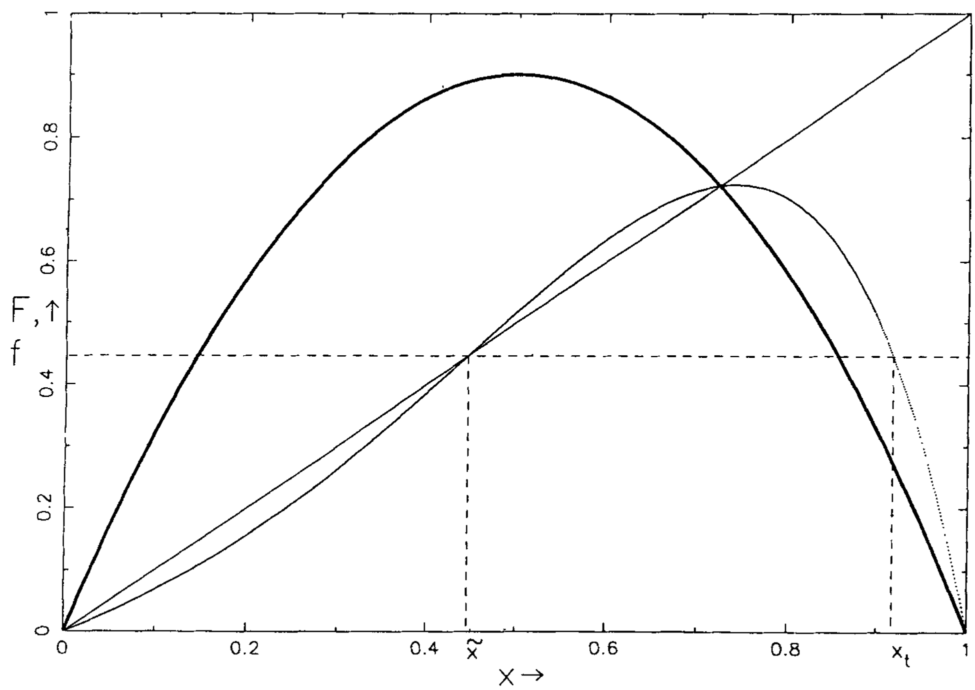

Figure 3. The basins of attraction of $x = 0$ and x_u^* for $\lambda^* = 0{\cdot}9$ and $\varepsilon = 2{\cdot}5$ for the logistic map, where the bold and dotted curves represent the functions F and f as functions of x respectively.

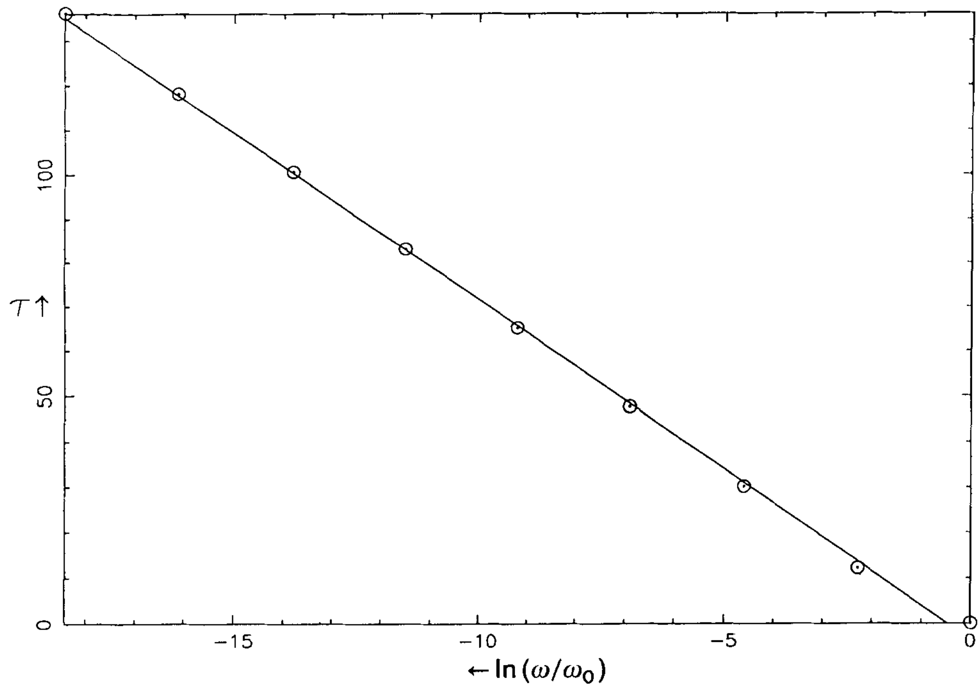

Figure 4. Plot of the transient time τ vs $\ln(\omega/\omega_0)$ for the logistic map with $\lambda^* = 0{\cdot}9$ and $\varepsilon = 2{\cdot}2$.

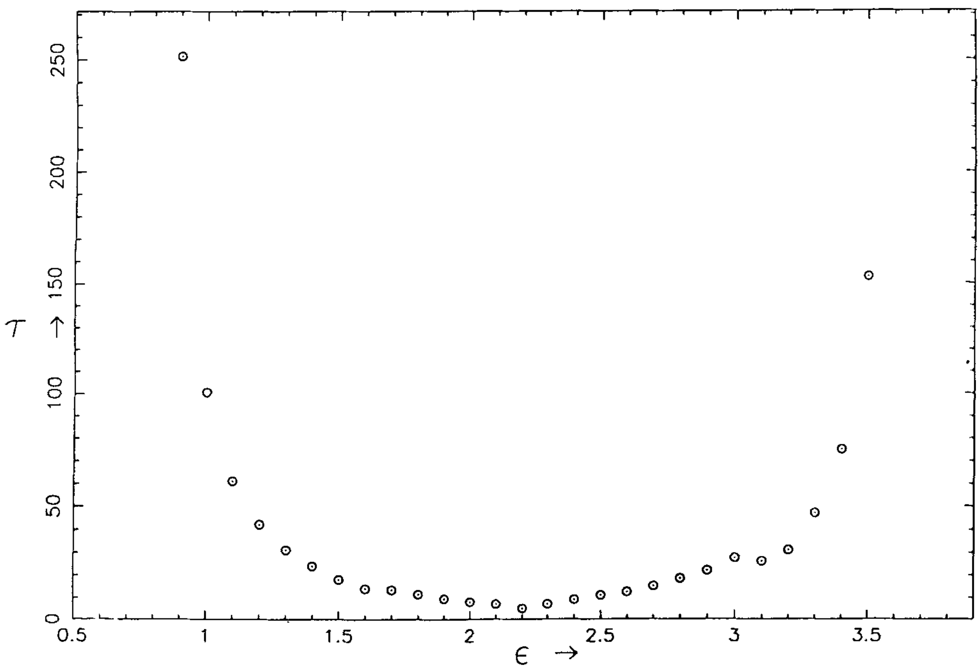

Figure 5. Plot of the transient time τ vs ε for the logistic map with $\lambda^* = 0{\cdot}9$ and $\omega = 10^{-7}$

3.1 *Behaviour of the logistic map while stabilizing period-2 fixed points*

The form of the control for stabilizing the logistic map to the period-2 unstable fixed points (x_1^*, x_2^*) for $\lambda^* > 0{\cdot}86238$, is given by

$$x_{n+1} = 4\lambda^* \exp(\varepsilon(x_n - x_1^*)(x_n - x_2^*))x_n(1 - x_n),$$
$$\equiv \mathscr{F}(\lambda^*, \varepsilon, x_n) \quad (5)$$

where $x_1^* = (1 + 4\lambda^* + \sqrt{(1 - 4\lambda^*)^2 - 4})/(8\lambda^*)$ and $x_2^* = (1 + 4\lambda^* - \sqrt{(1 - 4\lambda^*)^2 - 4})/(8\lambda^*)$. The variation of $\mathrm{d}\mathscr{F}^2/\mathrm{d}x|_{x_1^*, x_2^*}$ with λ^* for different values of ε is shown by circled lines in figure 6. The plot shows that the fixed points (x_1^*, x_2^*) corresponding to small λ^* can be stabilized using small values of ε while those corresponding to larger λ^* are stabilized by taking large values of ε. But large ε also tends to destabilize fixed points corresponding to small λ^* (the same situation as was observed with period-1 control). It also shows that there exists a range of values of $\varepsilon, \varepsilon'_{\min} < \varepsilon < \varepsilon'_{\max}$, for a given λ^*, for which (x_1^*, x_2^*) become the stable fixed points of the $\mathscr{F}^2$ map. Expressions for $\varepsilon'_{\min}(\lambda^*)$ and $\varepsilon'_{\max}(\lambda^*)$ can be obtained from the constraints $\mathrm{d}\mathscr{F}^2/\mathrm{d}x|_{x_1^*, x_2^*} > -1$ and $\mathrm{d}\mathscr{F}^2/\mathrm{d}x|_{x_1^*, x_2^*} < 1$ respectively. The range of values of ε for which (x_1^*, x_2^*) become stable fixed points of the $\mathscr{F}^2$ map is plotted against λ^* in figure 2.

It is also found numerically that for a given ε, there exists a critical value λ_{c1} of λ^* such that for $\lambda^* > \lambda_{c1}, \mathscr{F}^2(\lambda^*, \varepsilon, x)$ has four fixed points. These are $x = 0$, $x_1^*, \bar{x}, x_2^*$,

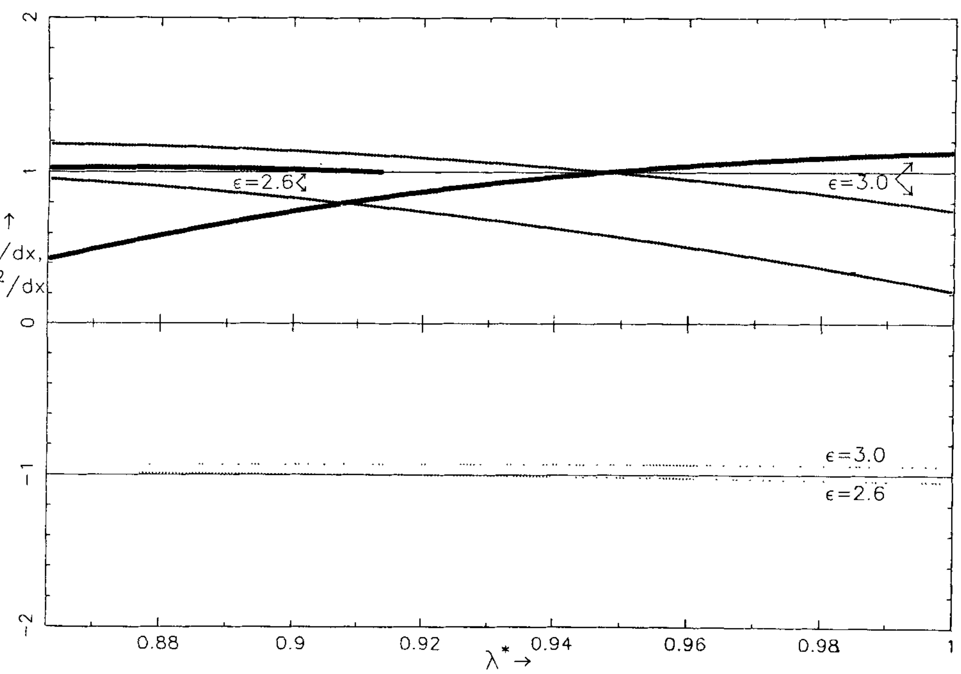

Figure 6. The variation of $\mathrm{d}\mathscr{F}^2/\mathrm{d}x$ for the logistic map evaluated at x_1' (or x_2') and x_1^* (or x_2^*) with λ^* is represented by bold and circled lines respectively for $\varepsilon = 2{\cdot}6$ and $3{\cdot}0$. The dotted line represents the variation of $\mathrm{d}\mathscr{F}/\mathrm{d}x$ evaluated at $x = \bar{x}$ for the same λ^* and ε's.

where x_1^*, x_2^* belong to the desired period-2 attractor, while the fixed points $x = 0$ and $\bar{x}$ are each of period 1. For $\lambda^* < \lambda_{c1}, \mathscr{F}^2(\lambda^*, \varepsilon, x)$ has two more fixed points x_1', x_2' in addition to the four fixed points mentioned above. Together x_1', x_2' form a new period-2 attractor. Variations of $\mathrm{d}\mathscr{F}^2/\mathrm{d}x|_{x_1^*, x_2^*}$, $\mathrm{d}\mathscr{F}^2/\mathrm{d}x|_{x_1', x_2'}$, $\mathrm{d}\mathscr{F}/\mathrm{d}x|_{\bar{x}}$ with λ^* for different values of ε are shown by circled, bold and dotted lines respectively in figure 6. Note that we are not studying the behaviour of $\mathrm{d}\mathscr{F}/\mathrm{d}x|_{x=0} = 4\lambda^* \exp(\varepsilon x_1^* x_2^*)$ which is always greater than one for $\lambda^* > 0{\cdot}86238$ and $\varepsilon > 0$, (which is required for stabilizing x_1^* and x_2^*) and hence $x = 0$ remains an unstable fixed point for $\varepsilon > 0$. For a given ε, the critical value λ_{c1} of λ^* can be obtained from the equation $\mathrm{d}\mathscr{F}/\mathrm{d}x|_{\bar{x}} = -1$. Figure 6 also shows that for a given ε and $\lambda^* < \lambda_{c1}$, either the desired period-2 attractor (x_1^*, x_2^*) or the new period-2 attractor (x_1', x_2') is stable. Another feature of period-2 control for the logistic map is that the function $\mathscr{F}^2(\lambda^*, \varepsilon, x)$ is not always bounded in the range $[0, 1]$ for $0 \leqslant x \leqslant 1$.

The transient τ shows a similar behaviour as in the previous section.

The control is found to work effectively for higher periods as well as for period cycles, which are created by tangent bifurcation, in the chaotic regime. We tested the control for stabilizing unstable fixed points of periods 3, 4 and 5 cycles and found it to be effective. The general features of the control remain the same.

4. Stabilization of arbitrary fixed points and the effect of noise

With exponential control it is possible to create new stable attractors which do not exist in the unmodulated system. This allows the stabilization of the system to arbitrary selected points. As an example we again take the logistic map. Suppose the aim is to stabilize the system to say a period-2 attractor (x_1, x_2) which does not exist in the unmodulated map, implying that x_1 and x_2 are not the fixed points of $F^2(\lambda, x)$. In order to achieve this the equations governing the dynamics in the presence of control are

$$\begin{aligned} x_{n+1} &= 4\lambda_1 \exp(\varepsilon(x_n - x_1)(x_n - x_2)) x_n (1 - x_n) \\ &\equiv g(x_n) \end{aligned}$$

and

$$\begin{aligned} x_{n+2} &= 4\lambda_2 \exp(\varepsilon(x_{n+1} - x_1)(x_{n+1} - x_2)) x_{n+1} (1 - x_{n+1}) \\ &\equiv g^2(x_n) \end{aligned} \qquad (6)$$

where λ_1 and λ_2 are given by

$$\lambda_1 = \frac{x_2}{4x_1(1 - x_1)}$$

and

$$\lambda_2 = \frac{x_1}{4x_2(1 - x_2)}.$$

The coordinates of the points lying in the region enclosed by the curves and the axes in figure 7 represent combinations of x_1 and x_2 for which $\mathrm{d}g^2/\mathrm{d}x|_{x_1, x_2}$ lies in $(-1, 1)$, implying that such combinations can be stabilized using the logistic map with exponential control.

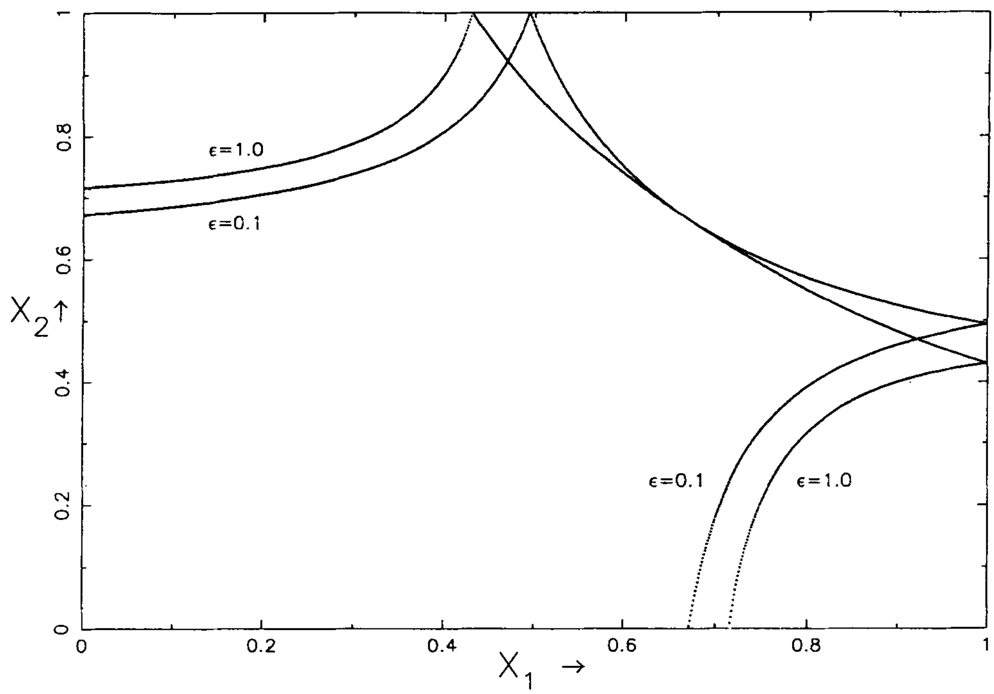

Figure 7. The coordinates of the points lying in the region enclosed by the curves and the axes represent the combinations of x_1 and x_2 which can be stabilized using the logistic map and the exponential control for $\varepsilon = 0{\cdot}1$ and $1{\cdot}0$.

This procedure can easily be extended to stabilize the system to trajectories of higher periods even when points on the trajectory do not correspond to either stable or unstable fixed points of the unmodulated map.

Next, we consider the issue of noise. We add term $\beta\delta_n$ to the right hand side of (4) of the modulated logistic map, where δ_n is a random quantity which has a Gaussian probability density with zero mean and unit standard deviation while β measures the noise level. We find that the exponential control given by (2) is capable of forcing the noisy modulated map to the desired fixed point x_u^* for the stiffness constant ε lying in the range $(\varepsilon_{min}, \varepsilon_{max})$, within an accuracy of the order of β. However, the basin of attraction of x_u^* gets affected by the value of β. Figure 8 shows the orbit plot, x_n vs n for 1000 iterates of the noisy modulated logistic map for $\mu = 0{\cdot}9$, $\varepsilon = 0{\cdot}9$ and $\beta = 0{\cdot}001$ starting from an initial state $x_0 = 0{\cdot}1$ and demonstrates the effectiveness of the control. We have also tested our control in the presence of noise which has a uniform distribution over [0, 1] and found the control to work effectively. The control is also found to stabilize desired higher period orbits as well as allowed combinations of arbitrary points in the noisy logistic map for the given range of values of ε.

We have also tested our control on the following 1-D maps and found it to work effectively with features similar to those described above:

(i) $F(\lambda, x) = \lambda \sin(\pi x)$

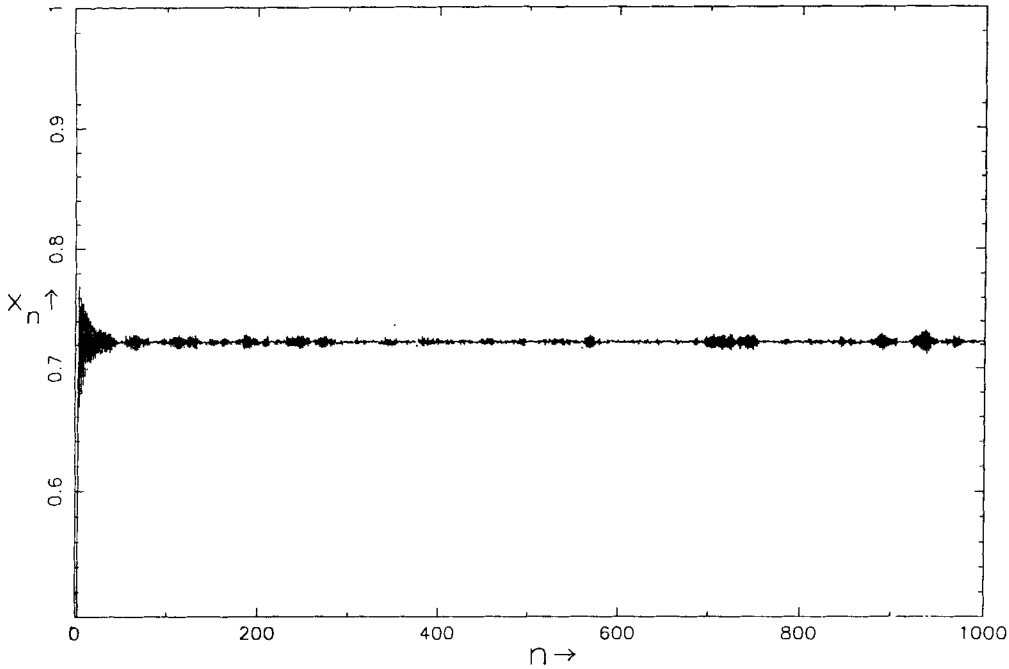

Figure 8. The orbit plot, x_n vs n for 1000 iterates of the modulated logistic map in the presence of noise having a Gaussian probability with zero mean and unit standard deviation, for $\mu = 0{\cdot}9$, $\varepsilon = 0{\cdot}9$ and $\beta = 0{\cdot}001$ starting from an initial state $x_0 = 0{\cdot}1$.

(ii) $F(\lambda, x) = \dfrac{4^{4/3}}{3}\lambda x(1 - x^3)$.

(iii) $F(\lambda, x) = 1 - \lambda|x|^2$.

5. Conclusion

The control suggested in this paper is a multiplicative nonlinear feedback control with which it is possible to stabilize unstable fixed points with periodicities that are (i) odd or (ii) multiples of 4 when created through period doubling route to chaos or (iii) with periodicities that are even but not multiples of 4 when created by tangent bifurcation for suitable ranges of the stiffness constant ε for 1 D maps which are of the form $F(\lambda, x) = \lambda G(x)$ or can be put in this form by a suitable transformation. Unstable fixed points of other periods may or may not be stabilized depending on the form of the function F and the values of the desired fixed points. The variation of the length of the transient τ as a function of ε for a given accuracy ω shows that τ is a decreasing function of ε. However there exists an optimum stiffness of control beyond which increasing ε can increase τ. The behaviour of the Lyapunov exponent as a function of ε is the same as that of τ. The optimum value of ε for which τ is minimum also corresponds to the most negative value, i.e. minimum, of Λ. In addition, it is also possible to stabilize arbitrary points, according to the required response of the system, which do not correspond to either stable or unstable fixed points of the unmodulated map. The control is found to work effectively even in the presence of noise.

Acknowledgements

We thank Inter University Centre for Astronomy and Astrophysics (IUCAA) for allowing the use of their computing facility. One of the authors (SDG) is grateful to Kirori Mal College, University of Delhi, for granting study leave.

References

[1] L M Pecora and T L Carroll, *Phys. Rev. Lett.* **64**, 821 (1990)
[2] L M Pecora and T L Carroll, *Phys. Rev. Lett.* **67**, 645 (1991)
[3] L M Pecora and T L Carroll, *Phys. Rev.* **A44**, 2374 (1991)
[4] E Ott, C Grebogi and J A Yorke, *Phys. Rev. Lett.* **64**, 1196 (1990)
[5] T Shinbrot, E Ott, C Grebogi and J A Yorke, *Phys. Rev. Lett.* **65**, 3215 (1990)
[6] W L Ditto, S N Rauseo and M L Spano, *Phys. Rev. Lett.* **65**, 3211 (1990)
[7] E R Hunt, *Phys. Rev. Lett.* **67**, 1953 (1991)
[8] C Reyl, L Flepp, R Badii and E Brun, *Phys. Rev.* **E47**, 267 (1993)
[9] J Güémez and M A Matías, *Phys. Lett.* **A181**, 29 (1993)
[10] P Cvitanović, *Universality in chaos* (Hilger, Bristol, 1984)
[11] K H Becker and M Dörfler, *Dynamical systems and fractals* (Cambridge, Cambridge, 1989)
[12] M C de Sousa Vieira, *J. Stat. Phys.* **53**, 1315 (1988)