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A new perturbative approach to the classical anharmonic oscillator

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Abstract. The periodic motion of the classical anharmonic oscillator characterized by the potential $V(x) = 1/2\,x^2 + \lambda/2k\,x^{2k}$ is considered. The period is first determined to all orders in λ in a perturbative series. Making use of this, the solution of the nonlinear equation of motion is then expressed in the form of a Fourier series. The Fourier coefficients are obtained by solving simple algebraic relations. Secular terms are inherently absent in this perturbative scheme. Explicit solution is presented for general k up to the second order, from which the Duffing and the sextic oscillator results follow as special cases.

Keywords. Classical AHO; secular term-free perturbative solution.

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1. Introduction

The time evolution of many classical systems is governed by nonlinear equations of motion that do not admit exact solutions. Anharmonic oscillators of various types comprise one class of such systems. In general, the motion of a nonlinear system can be determined only through some approximation method or other. A standard procedure is to treat the system as a linear system perturbed by the nonlinearity, and attempt to find a perturbative solution of the equation of motion. When this is done in a straightforward manner, what results is often a solution marred by secular terms, which render it unacceptable. In the case of the anharmonic oscillators, the secular terms make the solution nonperiodic and unbounded, whereas one knows on physical grounds that the motion is both bounded and periodic. Several techniques have been developed for eliminating secular terms from perturbative solutions [1]. In the Lindstedt–Poincare method, for instance, one introduces a scaled time variable, and expands both the solution and the scaling factor in separate perturbation series. Using the equation of motion, the unknowns are determined, order by order, in such a way that the secular term is cancelled out in every order. The other approximation methods proceed in a similar way, the secular terms being eliminated order by order.

A different approach is possible for a nonlinear system whose motion is known to be periodic. For such a system the solution can be expanded in a Fourier series with a fundamental period equal to the period T of oscillatory motion. The Fourier coefficients will then be functions of the perturbation parameter λ present in the problem. Expanding T in a power series in λ, one can determine perturbatively all the unknowns, order by order, as functions of λ. Secular terms are inherently absent in this scheme. A systematic perturbative formalism along these lines has been developed by Helleman and Montroll [2]. For the anharmonic oscillator (AHO) system, the H–M scheme amounts to determining simultaneously the frequency of the periodic motion and the displacement by writing separate perturbation series for each one. The coefficients in the two series are obtained from a nonlinear recurrence relation.

In the case of the AHO system a variation of the Fourier series method is possible, which differs significantly from the H–M method and offers considerable advantage. It is based on the fact that the period of classical motion of the AHO can be determined, quite independently of the detailed behaviour of the displacement as a function of time. This point does not seem to have been noticed in the literature. Once the period is determined, the original nonlinear differential equation reduces to a set of simple uncoupled linear algebraic equations for the Fourier coefficients. The rather involved analysis of Helleman and Montroll can be completely bypassed. The present paper demonstrates this in detail.

We consider the periodic motion of the anharmonic oscillator characterized by the potential energy

$$V(x) = \frac{1}{2}x^2 + \frac{1}{2k}\lambda x^{2k}, \qquad \lambda > 0$$

where k is a positive integer. The period is first computed to all orders in the parameter λ. We then obtain the solution $x(t)$ for general k explicitly up to the second order. Our perturbative solution reproduces (up to order λ^2) the known closed form results for the Duffing and the sextic anharmonic oscillators.

2. Perturbative expansion for period

The Hamiltonian for the general AHO is taken to be

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \frac{1}{2k}\lambda x^{2k}. \tag{1}$$

This leads to the nonlinear equation of motion

$$\ddot{x} + x + \lambda x^{2k-1} = 0. \tag{2}$$

We are interested in solving (2) subject to the initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{3}$$

Since H is conserved, there exists the first integral

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{2k}\lambda x^{2k} = E, \tag{4}$$

E being the total energy. From (4) it follows that the period T of oscillation is given by

$$T = \sqrt{2}\int_{-A}^{A} \frac{\mathrm{d}x}{[E - (1/2)x^2 - (\lambda/2k)x^{2k}]^{1/2}} \tag{5}$$

where $x = \pm A$ are the turning points defined by $E = V(x)$ [cf. (3)].

We convert the real integral above into a contour integral in the z plane of the form

$$T = \frac{1}{\sqrt{2}} \oint \frac{\mathrm{d}z}{[E - (1/2)z^2 - (\lambda/2k)z^{2k}]^{1/2}} \tag{6}$$

where the contour encloses only a branch cut joining $+A$ and $-A$, and no other singularity of the integrand. The branch of the square root chosen is that which is positive real on the upper lip of the cut; this necessitates that the contour be traversed clockwise. The above contour integral representation for the period proves to be particularly advantageous, as will be evident from what follows.

Making a perturbation expansion of the integrand in (6) in powers of λ, we obtain

$$\sqrt{2}\,T = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{\lambda}{2k}\right)^n I_n \tag{7}$$

where I_n is the integral

$$I_n = \oint \frac{\mathrm{d}z\, z^{2nk}}{(E - (1/2)z^2)^{n+1/2}} \tag{8}$$

In a different context we have considered similar contour integrals [3, 4]. To evaluate I_n, we rewrite it in the following form, with a simpler denominator for the integrand:

$$I_n = \frac{(-2)^n}{(2n-1)!!} \frac{\partial^n}{\partial E^n} \oint \frac{\mathrm{d}z\, z^{2nk}}{(E - (1/2)z^2)^{1/2}} \tag{9}$$

Now, the change of variable $z = \sqrt{2E}\,t$ enables the integrand's dependence on E to be factored out. We get

$$\oint \frac{\mathrm{d}z\, z^{2nk}}{(E - (1/2)z^2)^{1/2}} = \sqrt{2}(2E)^{nk} \oint \frac{\mathrm{d}t\, t^{2nk}}{(1 - t^2)^{1/2}}. \tag{10}$$

We observe that the integrand in (10) has only integrable singularities at $t = \pm 1$. Therefore we can evaluate it by compressing the contour on to the real axis. Thus

$$\oint \frac{\mathrm{d}t\, t^{2nk}}{(1 - t^2)^{1/2}} = 2 \int_{-1}^{1} \frac{\mathrm{d}t\, t^{2nk}}{(1 - t^2)^{1/2}} = 2B\left(nk + \frac{1}{2}, \frac{1}{2}\right), \tag{11}$$

where B is the Euler beta function defined by $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$. Substituting (10) and (11) in (9), and the resulting expression in (7), we get after some simplification

$$T = 2\pi \sum_{n=0}^{\infty} \binom{2nk}{nk} \binom{nk}{n} \left(-\frac{\lambda E^{k-1}}{2^k \cdot 2k}\right)^n. \tag{12}$$

We note at this point that the various manipulations performed on (8) are feasible because of its form as a closed contour integral in the complex plane. To derive the above expansion for T directly from the real integral (5) would entail far more labour.

The result (12) for the period is a perturbative series in λ whose coefficients are functions of the energy E, except for the case $k = 1$. For $k = 1$ the series can be summed, and yields, as it should, the value $T = 2\pi(1+\lambda)^{-1/2}$. The value of E is fixed by the initial conditions. For the conditions that we have chosen (see (3) above), the energy is the following function of the initial displacement A:

$$E = \frac{1}{2}A^2 + \frac{\lambda}{2k}A^{2k}. \tag{13}$$

Substituting for E in (12), we may obtain T in terms of λ and A. It is also possible to derive the following series for T in powers of A, by inserting (13) in (5) directly:

$$T = 2\pi \sum_{n=0}^{\infty} C_n \left(\frac{\lambda A^{2k-2}}{k}\right)^n \tag{14a}$$

where

$$C_n = \frac{1}{\pi}\binom{-\frac{1}{2}}{n}\int_{-1}^{+1}\frac{du(1-u^{2k})^n}{(1-u^2)^{n+1/2}} = \sum_{m=0}^{n}\frac{(-1)^m}{2^{2km}}\binom{n}{m}\binom{km}{n}\binom{2km}{km}. \tag{14b}$$

Introducing the frequency Ω, and expanding it in a perturbative series

$$\Omega = \frac{2\pi}{T} = 1 + \lambda\Omega_1 + \lambda^2\Omega_2 + \dots \tag{15a}$$

we note that the coefficients Ω_i will be functions of the parameter A. These can be determined using (12) or (14). The explicit expressions for Ω_1 and Ω_2 are

$$\Omega_1 = \frac{1}{2^{2k}}\binom{2k}{k}A^{2k-2} \tag{15b}$$

$$\Omega_2 = \frac{1}{2^{4k}}\left\{\binom{2k}{k}^2 + 2^{2k}\frac{(k-1)}{k}\binom{2k}{k} - \frac{2k-1}{k}\binom{4k}{2k}\right\}A^{4k-4}. \tag{15c}$$

The higher order coefficients Ω_i when needed can be calculated either from (12) and (13) or form (14).

We conclude this section with the following point. Had we chosen, instead of (1), the Hamiltonian

$$\tilde{H} = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2k}\alpha x^{2k},$$

the corresponding period $T(m,\omega,\alpha,E)$ would be related to the period defined by (5) through the equation

$$T(m,\omega,\alpha,E) = \frac{1}{\omega}\,T(1,1,\lambda,E) \tag{16a}$$

where

$$\lambda = \alpha/(m\omega^2)^k. \tag{16b}$$

This scaling law for the period is derived from (5) by a simple scaling of the integration variable. There is no loss of generality therefore in choosing for the Hamiltonian the form given in (1). It may be noted that the scaling law for T regarded as a function of A is $T(m,\omega,\alpha,A) = \omega^{-1}\,T(1,1,\alpha/m\omega^2,A)$.

3. First order solution

The solution to the equation of motion must be a periodic function with period T. Since in the absence of the anharmonic term the solution satisfying the initial conditions (3) is $A\cos t$, we write for the first order solution $x(t) = A\cos\Omega t + \lambda f(t)$. The function f itself must also have the same period, and can therefore be expressed as a Fourier series. The initial velocity having been taken as zero, the series for f will not contain any sine terms. Writing $f(t) = \Sigma C_n \cos n\Omega t$, we can deduce from the structure of the equation of motion that the series for $f(t)$ contains only a finite number of odd cosine terms, and no even cosine terms. We are thus led to write

$$x(t) = A\cos\Omega t + \lambda\sum_{n=0}^{k-1} a_n \cos(2n+1)\Omega t + O(\lambda^2). \tag{17}$$

It is necessary that the coefficients a_n obey the relation

$$\sum_{n=0}^{k-1} a_n = 0 \tag{18}$$

in order that the initial condition $x(0) = A$ be satisfied. We shall see that it is always possible to ensure condition (18).

Substituting (17) into (2), using the expansion $\Omega^2 = 1 + 2\lambda\Omega_1 + O(\lambda^2)$, and retaining only terms of order λ, we arrive at the equation

$$\sum_{n=0}^{k-1} a_n[1-(2n+1)^2]\cos(2n+1)\Omega t$$
$$= 2\Omega_1 A\cos\Omega t - \frac{A^{2k-1}}{2^{2k-2}}\sum_{n=0}^{k-1}\binom{2k-1}{k-1-n}\cos(2n+1)\Omega t. \tag{19}$$

In obtaining (19) we have made use of the standard formula

$$\cos^{2k-1}\theta = \frac{1}{2^{2k-2}}\sum_{n=0}^{k-1}\binom{2k-1}{k-1-n}\cos(2n+1)\theta.$$

The coefficients a_n can be determined from (19). We observe that on the lhs of (19), the coefficient of $\cos\Omega t$ vanishes. Therefore, for consistency, the rhs also must be free of $\cos\Omega t$ term. Indeed it is so, as can be easily checked. The coefficient of $\cos\Omega t$ term on the rhs is

$$2A\left\{\Omega_1 - \frac{A^{2k-2}}{2^{2k-1}}\binom{2k-1}{k-1}\right\},$$

and this vanishes because of (15b). Therefore, a_0 in (17) is arbitrary. All the other a_n's are determined uniquely by (19), with the result

$$a_n = \frac{1}{(2n+1)^2-1}\frac{A^{2k-1}}{2^{2k-2}}\binom{2k-1}{k-1-n}, \quad n>0. \tag{20}$$

It is now evident that in order to satisfy (18) we should set

$$a_0 = \sum_{n=1}^{k-1}\frac{1}{1-(2n+1)^2}\frac{A^{2k-1}}{2^{2k-2}}\binom{2k-1}{k-1-n}.$$

The summation on the rhs can be carried out using the identity

$$\sum_{m=0}^{k-2}\binom{2k-1}{m}\frac{1}{(k-m)(k-m-1)}=\frac{1}{2k}\left[(k+1)\binom{2k}{k}-2^{2k}\right]$$

and we get

$$a_0=\frac{A^{2k-1}}{2k\,2^{2k}}\left[2^{2k}-(k+1)\binom{2k}{k}\right]. \tag{21}$$

We thus have the solution to first order

$$x(t)=A\cos\Omega t+\lambda\sum_{n=0}^{k-1}a_n\cos(2n+1)\Omega t, \tag{22}$$

with a_0 given by (21) and $a_n, n>0$, by (20).

4. Solution to second order

To determine the solution to the second order in λ, we make the following ansatz

$$x(t)=A\cos\Omega t+\lambda\sum_{n=0}^{k-1}a_n\cos(2n+1)\Omega t$$
$$+\lambda^2\sum_{n=0}^{2(k-1)}b_n\cos(2n+1)\Omega t+O(\lambda^3) \tag{23}$$

where the first order coefficients a_n have already been obtained. We observe first that

$$\begin{aligned}\ddot{x}+x&=[-2\lambda\Omega_1-\lambda^2(\Omega_1^2+2\Omega_2)]A\cos\Omega t\\&+\lambda\sum a_n[1-(2n+1)^2-2\lambda\Omega_1(2n+1)^2]\cos(2n+1)\Omega t\\&+\lambda^2\sum b_n[1-(2n+1)^2]\cos(2n+1)\Omega t\end{aligned}$$

where we have used the expansion $\Omega^2=1+2\lambda\Omega_1+\lambda^2(\Omega_1^2+2\Omega_2)$ and retained terms up to order λ^2. The nonlinear term yields, to this order,

$$\lambda x^{2k-1}=\lambda A^{2k-1}\cos^{2k-1}\Omega t+$$
$$\lambda^2(2k-1)A^{2k-2}\cos^{2k-2}\Omega t.\sum a_n\cos(2n+1)\Omega t.$$

Picking out the λ^2 terms from the above we get

$$\begin{aligned}&\sum_{n=0}^{2k-2}b_n[1-(2n+1)^2]\cos(2n+1)\Omega t\\&=(\Omega_1^2+2\Omega_2)A\cos\Omega t+2\Omega_1\sum_{n=0}^{k-1}a_n(2n+1)^2\cos(2n+1)\Omega t\\&-(2k-1)A^{2k-2}\sum_{n=0}^{k-1}a_n\cos(2n+1)\Omega t\cos^{2k-2}\Omega t.\end{aligned} \tag{24}$$

Using the formula

$$\cos^{2n}\theta = \frac{1}{2^{2n}}\left[\binom{2n}{n} + \sum_{m=0}^{n-1} 2\binom{2n}{m}\cos 2(n-m)\theta\right]$$

the last term on the rhs of (24) can be expressed as sum of cosines. After some algebra we get

$$\sum_{n=0}^{2k-2} b_n[1-(2n+1)^2]\cos(2n+1)\Omega t$$

$$= C\cos\Omega t + 2\Omega_1 \sum_{n=1}^{k-1} a_n[(2n+1)^2 - k]\cos(2n+1)\Omega t$$

$$- \frac{(2k-1)A^{2k-2}}{2^{2k-2}}\left\{\sum_{n=1}^{2k-2} f_n \cos(2n+1)\Omega t + \sum_{n=1}^{k-2} g_n \cos(2n+1)\Omega t\right\} \tag{25a}$$

where

$$f_n = \sum_{m=0}^{n-1} a_m \binom{2k-2}{k-1-n+m}, \tag{25b}$$

$$g_n = \sum_{m=n+1}^{n+k-1} a_m \binom{2k-2}{k-1+n-m} + \sum_{m=0}^{k-2-n} a_m \binom{2k-2}{k-2-n-m}, \tag{25c}$$

$$C = A\left(\Omega_1^2 + 2\Omega_2 + \frac{(k^2-1)}{k}\Omega_1^2\right) + 2\Omega_1\frac{(1-k)}{k}A^{2k-1}$$
$$- \frac{(2k-1)A^{2k-2}}{2^{2k-2}} \sum_{m=0}^{k-2} a_{m+1}\binom{2k-1}{k-2-m}. \tag{26}$$

The second order coefficients b_n are to be determined from (25), the rhs of which involves only known quantities. As in the first order, the coefficient b_0 on the lhs is zero. On the rhs, C is found to vanish on substituting the known values of a_n, Ω_1 and Ω_2 in (26). This makes b_0 arbitrary. The other b_n's are determined uniquely by (25). Taking $a_n = 0$ for $n \geqslant k$ (a_n given by (20) vanishes in fact for $n \geqslant k$), the sum over n on the rhs of (25a) can be extended from $k-1$ to $2k-2$. Putting in the value of Ω_1 and rearranging terms, we get finally the expression

$$b_n = \frac{A^{2k-2}}{2^{2k}n(n+1)}\left\{-\frac{(2n+1)^2}{2}\binom{2k}{k}a_n + (2k-1)\sum_{m=0}^{n+k-1}\binom{2k-2}{k-1-n+m}a_m\right.$$
$$\left. + (2k-1)\sum_{m=0}^{k-2-n}\binom{2k-2}{k-2-n-m}a_m\right\}, \qquad n > 0. \tag{27}$$

As in the case of a_0 in first order, the arbitrary coefficient b_0 is fixed by the requirement that $\Sigma_{n=0}^{2k-2} b_n = 0$ thus ensuring that the initial condition $x(0) = A$ is satisfied in the second order also.

5. General solution

The solution derived above corresponds to a particular set of initial conditions, namely arbitrary displacement and zero velocity. The analysis of the AHO is simplest with these conditions. It is possible to obtain from this particular solution the general solution corresponding to arbitrary initial position and velocity. To this end, we note that the equation of motion (2) is invariant under time translations. Therefore, if we replace Ωt in (23) by $\Omega t + \phi$, we still have a solution, but with the change that A is no longer identifiable as the initial displacement $x(0)$. Explicitly, the new solution

$$\tilde{x}(t) = A\cos(\Omega t + \phi) + \lambda \sum a_n \cos(2n+1)(\Omega t + \phi) + \lambda^2 \sum b_n \cos(2n+1)(\Omega t + \phi) + O(\lambda^3) \tag{28}$$

is seen to contain two arbitrary parameters A and ϕ, which serve to accommodate arbitrary initial conditions. It should be noted that the frequency Ω occurring in (28) does not change: it is still the same function of the parameter A. This is due to the fact that the frequency of the AHO is determined by the value of E, which is a constant of the motion, and for a given E, there is always an instant at which the velocity vanishes. For arbitrary initial values x_0 and v_0, one may use the form (28) and determine A and ϕ in terms of x_0 and v_0, which is not a simple task. Alternatively, one can use the solution (23) but with the time being reckoned from the instant at which the velocity vanishes.

6. Special cases

It is instructive to consider two special cases in which the equation of motion can be solved in closed form. These are the Duffing oscillator ($k = 2$) and the sextic AHO ($k = 3$).

In the Duffing case, our perturbative solution is

$$x(t) = A\cos\Omega t + \frac{\lambda A^3}{32}(\cos 3\Omega t - \cos\Omega t) + \frac{\lambda^2 A^5}{1024}(\cos 5\Omega t - 24\cos 3\Omega t + 23\cos\Omega t) + O(\lambda^3) \tag{29}$$

with the frequency given by

$$\Omega = 1 + \frac{3}{8}\lambda A^2 - \frac{21}{256}\lambda^2 A^4 + O(\lambda^3). \tag{30}$$

The exact solution satisfying the same initial conditions as (29) can be expressed as

$$x(t) = A\, cn(\sqrt{1 + \lambda A^2}\, t, k) \tag{31}$$

where cn is the Jacobian elliptic function whose modulus k is given by

$$k^2 = \frac{\lambda A^2}{2}(1 + \lambda A^2)^{-1}. \tag{32}$$

The exact period of oscillation is

$$T = 4K(k)(1 + \lambda A^2)^{-1/2} \tag{33}$$

where $K(k)$ is the complete elliptic integral of the first kind. The expression (30) coincides up to order λ^2 with the expansion for Ω in powers of λ obtained from (33). By making use of the standard Fourier series expansion of the *cn* function [5] we can show that (29) is the same as (31) up to order λ^2.

Similar results hold also in the case of the sextic AHO. The exact solution for this case can be written in the form

$$x(t) = \frac{A\, cn\, u}{dn\, u}\left[\frac{1 + dn\, 2u}{1 + \gamma + (1-\gamma) cn\, 2u}\right]^{1/2} \tag{34}$$

where

$$u = \sqrt{\alpha\beta}\, t, \quad \alpha = (1 + \lambda A^4)^{1/2}, \quad \beta = \left(1 + \frac{\lambda A^4}{3}\right)^{1/2}, \quad \gamma = \frac{\alpha}{\beta} \tag{35}$$

and the modulus of the elliptic functions *cn* and *dn* is

$$k^2 = \frac{1}{2}\left[1 - \frac{1}{\alpha\beta}\left(1 + \frac{1}{2}\lambda A^4\right)\right] \tag{36}$$

The frequency of this periodic solution is

$$\Omega = \pi\sqrt{\alpha\beta}/2K(k) \tag{37}$$

Expanding the solution (34) in powers of λ, we get

$$\begin{aligned} x(t) = {} & A\cos\Omega t + \frac{\lambda A^5}{384}(\cos 5\Omega t + 15\cos 3\Omega t - 16\cos\Omega t) \\ & + \frac{\lambda^2 A^9}{294912}(3\cos 9\Omega t + 95\cos 7\Omega t - 7680\cos 3\Omega t + 7582\cos\Omega t) \\ & + O(\lambda^3) \end{aligned} \tag{38}$$

which is precisely the solution generated by our second order formula (23). Similarly, we get from (37) the expansion

$$\Omega = 1 + \frac{5}{16}\lambda A^4 - \frac{215}{3072}\lambda^2 A^8 + O(\lambda^3). \tag{39}$$

which coincides with the result derived from (15).

7. Discussion

We have presented above explicit classical solution to the general anharmonic oscillator up to the second order in the anharmonicity. Novel features of our perturbative approach are the prior determination of the period T of classical motion (to all orders in the nonlinearity parameter λ) and its subsequent use to develop a small-coupling expansion for the solution which is free of secular terms. In usual perturbative treatments, the frequency Ω and the displacement $x(t)$ are found simultaneously in every order of the approximation, which makes the analysis of the general AHO rather involved. In our method we exploit the fact that the AHO has a well defined period which can be determined independently, without any references to the displacement $x(t)$ (except for the initial conditions). With the frequency of classical motion Ω in hand, physical reasoning suggests a perturbative ansatz for $x(t)$ as a Fourier series with Ω as the fundamental frequency. As we have seen, the Fourier coefficients can be systematically determined order by order, the unknown coefficients in any given order satisfying uncoupled, linear algebraic equations involving coefficients of the preceding orders. In every order one coefficient is left undetermined and arbitrary by the equations. This gives us the freedom to ensure that the initial condition $x(0) = A$ is satisfied in every order of the approximation.

The Fourier series ansatz for the displacement $x(t)$ guarantees the absence of secular terms in the perturbation expansion, as noted also by Helleman and Montroll [2] earlier. For the ansatz to work, it is essential that there should be no $\cos \Omega t$ terms on the rhs of the first order equation (19) and the second order equation (25). The use of the correct Ω ensures the absence of these unwanted terms. We note that if $\cos \Omega t$ term were present on the rhs of (19) or (25), the ansatz would fail, implying that the solution is then not periodic, which is tantamount to the presence of secular terms in the solution.

Our analysis of the general AHO system is far simpler than that presented by Helleman and Montroll. As noted earlier, they expand both $x(t)$ and Ω in series in powers of λ, and obtain a nonlinear recurrence relation for the coefficients. It would be a cumbersome task to extract the coefficients from their recurrence relation. These authors have not given any explicit expressions either for the frequency or for the displacement.

In this work our analysis does not go beyond order λ^2, as it is rarely that one would need to go further. There is however no difficulty in principle in including higher orders. Nor is there any difficulty in adapting the method to AHO's with anharmonic terms which are even polynomials of degree $2k$.

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