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Some exact solutions in Bianchi VI_0 string cosmology

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Abstract. Following the techniques used by Letelier and Stachel some new physically relevant explicit Bianchi VI_0 solutions of string cosmology with magnetic field are reported. They include two models describing distributions of Takabayashi strings and geometric strings respectively.

Keywords. Exact solutions; string cosmology.

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1. Introduction

The study of cosmic strings has received considerable interest. Existence of large scale networks of strings in the early universe is confirmed by the present day observations of the universe [1]. The cosmic strings are believed to give rise to density perturbations leading to galaxy formations [2]. Cosmic strings possess stress-energy and are coupled to the gravitational field. The gravitational effects of such strings have been studied by several authors (Vilenkin [3], Gott [4] and Garfinkle [5]).

Letelier [6] and Stachel [7] have developed general relativistic treatment of strings. Letelier [8] obtained relativistic cosmological models in Bianchi I and Kantowski-Sachs space-times adopting the energy-momentum tensor as given by

$$T_{ik} = \rho u_i u_k - \lambda w_i w_k, \tag{1}$$

$$u_i u^i = -w_i w^i = 1, \quad u^i w_i = 0,$$

as the source term in Einstein's field equations

$$R_{ik} - (\tfrac{1}{2}) R g_{ik} = -8\pi T_{ik}. \tag{2}$$

In (1), ρ represents energy density for a cloud of strings with particles attached to them, λ denotes the string tension density, the space-like vector w_i represents the direction of the string and the unit time-like vector u_i is the flow vector of matter.

There is no direct evidence of strings in the present-day universe. Accordingly, cosmological models evolving from the string dominated era and ending up in particle dominated era seem to be of physical relevance. In accordance with Melvin's argument [9] for a large part of the history of evolution of the universe, the matter was in a highly ionized state, smoothly coupled with field, subsequently forming neutral matter as a result of universe expansion. Hence presence of magnetic field in a string-dust universe is not beyond reality. Relativistic models in the context of Bianchi type II,

VI_0, VIII and IX space-times describing cosmological distributions of massive strings have been discussed by Krori *et al* [10]. Banerjee *et al* [11] have studied the effects of presence of magnetic field in the context of the string cosmological models in the Bianchi type I space-times. A class of solutions in the context of Bianchi VI_0 string cosmology has been obtained by Chakraborty [12]. In this paper we have reported some new exact solutions of string cosmology with and without magnetic fields respectively in the context of the Bianchi type VI_0 space-times. Further these solutions have been compared with those obtained by Chakraborty and their physical relevance discussed.

2. Field equations

The metric of Bianchi VI_0 space-time has the general form

$$ds^2 = A^2(t)(dt^2 - dx^2) - B^2(t)e^{2mx}dy^2 - C^2(t)e^{-2mx}dz^2, \tag{3}$$

where m is a constant. The energy-momentum tensor for a cloud of string dust with magnetic field along string direction, given by

$$T_{ik} = \rho u_i u_k - \lambda w_i w_k + \frac{1}{4\pi}\left[-g^{lm}F_{il}F_{km} + \frac{1}{4}g_{ik}F_{lm}F^{lm}\right], \tag{4}$$

where u_i and w_i satisfy the conditions stated in (1) and F_{ik} is the Maxwell's stress tensor, is taken as the source term in (2). We choose x-direction as the direction of strings along which the magnetic field is assumed to be present. In comoving coordinates

$$u^i = (0,0,0,1/A), \quad w^i = (1/A,0,0,0). \tag{5}$$

Maxwell's equations $F_{(i,k;l)} = 0$ and $(F^{ik}\sqrt{-g})_{,k} = 0$, imply that

$$F_{23} = K(\text{constant}) \tag{6}$$

is the only surviving component of F_{ik}. Equations (5) and (6) on substitution in (4) imply that $T_{14} = 0$. The field equations (2) consequently require that $R_{14} = 0$. In the context of the metric (3) this implies

$$R_{14} = -m\left[\frac{B'}{B} - k\frac{C'}{C}\right] = 0. \tag{7}$$

Here and in what follows a prime indicates a differentiation wth respect to t. For Bianchi VI_0 space-times $m = 0$ and subsequently equation (7) implies that $B = C$.

Einstein's field equations (2) then lead to

$$\frac{1}{A^2}\left[\frac{A''}{A} - \frac{A'^2}{A^2} + 2\frac{A'B'}{AB} + \frac{B''}{B} + \frac{B'^2}{B^2} - 2m^2\right] = 8\pi\rho, \tag{8}$$

$$\frac{1}{A^2}\left[-\frac{A''}{A} + \frac{A'^2}{A^2} - 2\frac{A'B'}{AB} + \frac{B''}{B} + \frac{B'^2}{B^2} - 2m^2 - 2\frac{K^2A^2}{B^4}\right] = 8\pi\lambda, \tag{9}$$

$$\left[\frac{A''}{A} - \frac{A'^2}{A^2} + \frac{B''}{B} - m^2 + \frac{K^2A^2}{B^4}\right] = 0. \tag{10}$$

Let ρ_p denote the particle energy density of the configuration so that

$$\rho = \rho_p + \lambda. \tag{11}$$

Equations (8) and (9) accordingly give

$$8\pi\rho_p = \frac{1}{A^2}\left[2\frac{A''}{A} - 2\frac{A'^2}{A^2} + 4\frac{A'B'}{AB} - 4m^2 + 2\frac{K^2A^2}{B^4}\right]. \tag{12}$$

The velocity field u^i, described by (5) is irrotational. The expansion parameter θ and the shear parameter σ^2 for the metric (3) with $B = C$ are given by

$$\theta = \frac{1}{A}\left[\frac{A'}{A} + \frac{2B'}{B}\right], \tag{13}$$

$$\sigma^2 = \frac{1}{3A^2}\left[\frac{A'}{A} - \frac{B'}{B}\right]^2 \tag{14}$$

Raychaudhuri equation [13] in view of the field equations (8)–(10) can be written as

$$\theta' = -\frac{1}{\theta^2} - 2\sigma^2 - 4\pi\rho_p - \frac{K^2}{B^4}, \tag{15}$$

where $R_{ik}u^iu^k = (-K^2/B^4) = -4\pi\rho_p$. The energy conditions imply that $\rho_p \geqslant 0$. The above expression accordingly indicates that the gravitational collapse of the system cannot be avoided in the presence of strings and the magnetic field. The energy conditions do not put any restriction on the sign of the string tension density λ.

3. Explicit solutions

The field equations (8)–(10) constitute a system of three equations with four unknown parameters ρ, λ, A and B. An additional constraint equation relating these parameters is necessary to obtain explicit solutions of the system. Equation (10) cannot be integrated in general unless $A(t)$ or $B(t)$ is specified. Assuming $A(t) = 1$, Chakraborty [12] obtained a class of Bianchi VI_0 solutions of these equations with and without magnetic field. Several assumptions can be made in this regard and explicit solutions can be obtained. We discuss here two fairly simple and physically relevant solutions of this system which are distinct from those reported by Chakraborty.

Solution 1

The Bianchi VI_0 space-time with metric

$$\begin{aligned} ds^2 = {} & (A_0)^2 e^{4(m^2-K^2)^{1/2}t}(dt^2 - dx^2) \\ & - A_0 e^{2(m^2-K^2)^{1/2}t}(e^{2mx}dy^2 - e^{-2mx}dz^2) \end{aligned} \tag{16}$$

where A_0 is an arbitrary constant of integration, is a solution of equation (10). The physical and the kinematical parameters corresponding to this model are

$$8\pi\rho = \frac{2}{(A_0)^2}(m^2 - K^2)e^{-4(m^2-K^2)^{1/2}t},$$

$$8\pi\lambda = -\frac{2m^2}{(A_0)^2} e^{-4(m^2-K^2)^{1/2}t},$$

$$8\pi\rho_p = \frac{2(2m^2-3K^2)}{(A_0)^2} e^{-4(m^2-K^2)^{1/2}t},$$

$$\theta = \frac{4(m^2-K^2)^{1/2}}{A_0} e^{-2(m^2-K^2)^{1/2}t},$$

$$\sigma^2 = \frac{(m^2-K^2)^{1/2}}{3(A_0)^2} e^{-4(m^2-K^2)^{1/2}t}. \tag{17}$$

It can be observed that the energy conditions $\rho > 0$, $\rho_p > 0$ are satisfied when $K^2 < 2m^2/3$. The expansion scalar θ is always positive. The Bianchi VI_0 space-time of metric (16) describes an expanding string cosmological model without singularity. For $K = 0$, the model degenerates into a Bianchi VI_0 model of string cosmology without magnetic field.

Solution 2

When $m^2 = K^2$, a solution of (10) can be obtained in the form $A = B^2 = (\alpha t)^6$ without any loss of generality, where α is an arbitrary constant. The Bianchi VI_0 space-time of this solution has the metric

$$ds^2 = (\alpha t)^{12}(dt^2 - dx^2) - (\alpha t)^6(e^{2Kx}dy^2 + e^{-2Kx}dz^2). \tag{18}$$

In this model the Bianchi VI_0 nature of the space-time is directly linked with the presence of the magnetic field. In the absence of magnetic field i.e., $K = 0$ the metric (18) becomes of Bianchi I type, representing a string model without magnetic field. The physical and the kinematical parameters of this model have the explicit expressions

$$8\pi\rho = \frac{45 - 2K^2t^2}{\alpha^{12}t^{14}}, \quad 8\pi\lambda = -\frac{15}{\alpha^{12}t^{14}}, \quad 8\pi\rho_p = \frac{60 - 2K^2t^2}{\alpha^{12}t^{14}}, \tag{19}$$

$$\theta = \frac{12}{\alpha^6 t^7}, \quad \sigma^2 = \frac{3}{\alpha^{12}t^{14}}. \tag{20}$$

The model begins with a singularity at $t = 0$ and complies with the requirement $\rho > 0$ inherent in the energy conditions until the epoch $t = \frac{3}{K}\sqrt{\frac{5}{2}}$, where $\rho = 0$. The expressions further indicate that the model remains physically significant for weak magnetic fields i.e. $K \ll 1$. The string tension density is always negative.

4. Explicit solutions in the absence of magnetic field

In the absence of the magnetic field, the field equations (8)–(10) become

$$\left[\frac{A''}{A} - \frac{A'^2}{A^2} + 2\frac{A'B'}{AB} + \frac{B''}{B} + \frac{B'^2}{B^2} - 2m^2\right] = 8\pi\rho A^2 \tag{21}$$

$$\left[-\frac{A''}{A}+\frac{A'^2}{A^2}-2\frac{A'B'}{AB}+\frac{B''}{B}+\frac{B'^2}{B^2}+2m^2\right]=8\pi\lambda A^2, \tag{22}$$

$$\left[\frac{A''}{A}-\frac{A'^2}{A^2}+\frac{B''}{B}-m^2\right]=0, \tag{23}$$

In this case also explicit solutions of (23) can be obtained by assuming an additional relation between the parameters ρ, λ, A and B. We report here two solutions which lead to physically relevant models of string cosmology.

We split equation (23) so that either

$$\frac{A''}{A}-\frac{A'^2}{A^2}=m^2$$

and

$$B''=0 \tag{24}$$

or

$$\frac{A''}{A}-\frac{A'^2}{A^2}=0$$

and

$$B''-m^2B=0. \tag{25}$$

Equations (24) admit the solution

$$A=\exp(\tfrac{1}{2}m^2t^2+at),\ B=\alpha t+\beta, \tag{26}$$

where a, α and β are constants of integration. Without any loss of generality we can choose $\beta=0$ and $\alpha=1$, so that the metric corresponding to this solution reads

$$\mathrm{d}s^2=\exp(m^2t^2+2at)[\mathrm{d}t^2-\mathrm{d}x^2]-t^2(e^{2mx}\mathrm{d}y^2+e^{-2mx}\mathrm{d}z^2). \tag{27}$$

The space-time is singularity-free. The physical and the kinematical parameters associated with this model are obtained as

$$8\pi\rho=\frac{(1+2at+m^2t^2)}{t^2}\exp(-m^2t^2-2at),$$

$$8\pi\lambda=\frac{(1-2at-m^2t^2)}{t^2}\exp(-m^2t^2-2at),$$

$$8\pi\rho_p=\frac{2(m^2t+2a)}{t}\exp(-m^2t^2-2at),$$

$$\theta=\frac{(2+at+m^2t^2)}{t}\exp(-m^2t^2-2at)/2$$

$$\sigma^2=\frac{(1-at-m^2t^2)}{3t^2}\exp(-m^2t^2-2at). \tag{28}$$

These expressions clearly indicate that all the parameters diverge as $t\to 0$. Accordingly the space-time of this model evolves from the singularity at $t=0$ undergoing expansion. If $a\geqslant 0$, the conditions $\rho>0$, $\rho_p>0$ are satisfied at all epochs.

Equation (25) leads to the solution

$$A = e^{\alpha t + \beta}, \; B = C_1 e^{mt} + C_2 e^{-mt}, \tag{29}$$

where α, β, C_1 and C_2 are constants of integration. Without any loss of generality we can choose $\beta = 0$ and $C_1 = 1$. It is found that if $C_2 = 0$, the above solution leads to a physically relevant model with the metric

$$ds^2 = e^{2\alpha t}[dt^2 - dx^2) - e^{2m(t+x)}dy^2 - e^{2m(t-x)}dz^2. \tag{30}$$

The space-time metric is singularity-free. The physical and the kinematical parameters in this case have the explicit expressions

$$\begin{aligned} &\frac{\rho}{m(2\alpha - m)} = \frac{\lambda}{m(3m - 2\alpha)} = \frac{\rho_p}{4m(\alpha - m)} = e^{-2\alpha t}, \\ &\theta = (\alpha + 2m)e^{-\alpha t}, \\ &\quad \sigma^2 = \frac{(\alpha - m)^2}{3} e^{-2\alpha t}. \end{aligned} \tag{31}$$

The energy density ρ and the particle energy density ρ_p are positive for $m < \alpha$. The string dust distribution has the equation of state

$$\rho = (1 + W)\lambda \tag{32}$$

with $W = 4(m - \alpha)/(2\alpha - 3m)$, characterizing Takabayashi strings [14] for which $W > 0$. This implies the restriction $2\alpha < 3m$. For $\alpha = m$ the kinematical parameter $\sigma = 0$. Further $\rho_p = 0$ and

$$8\pi\rho = 8\pi\lambda = m^2 e^{-2\alpha t}.$$

The model subsequently describes a cosmological distribution of geometric strings characterized by equation of state $\rho = \lambda$.

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References

[1] T W S Kible, *J. Phys.* **A9**, 1387 (1976)
[2] Ya B Zeldovich, *Mon. Not. R. Astron. Soc.* **192**, 663 (1980)
[3] A Vilenkin, *Phys. Rev.* **D24**, 2082 (1981)
[4] J R Gott, *Astrophys. J.* **288**, 422 (1985)
[5] D Garfinkle, *Phys. Rev.* **D32**, 1323 (1985)
[6] P S Letellier, *Phys. Rev.* **D20**, 1294 (1979)
[7] J Stachel, *Phys. Rev.* **D21**, 2171 (1980)
[8] P S Letellier, *Phys. Rev.* **D28**, 2414 (1983)
[9] M A Melvin, *Ann. N.Y. Acad. Sci.* **262**, 253 (1975)
[10] K D Krori, T Chaudhuri, C R Mahanta and A Mazumdar, *Gen. Relativ. Gravit.* **22**, 123 (1990)

[11] A Banerjee, A K Sanyal and S Chakraborty, *Pramana – J. Phys.* **34**, 1 (1990)
[12] S Chakraborty, *Indian J. Pure Appl. Phys.* **29**, 31 (1991)
[13] A K Raychaudhuri, (Clarendon Press, Oxford, 1979)
[14] T Takabayashi, edited by M Flato (Holland, Reidel, Dordrecht)