PRAMANA
— journal of
physics

© Printed in India

Vol. 42, No. 6,
June 1994
pp. 467–472

On a time-dependent system of noncentral anharmonic oscillators in two dimensions

R S KAUSHAL
Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India

MS received 6 January 1994; revised 29 April 1994

Abstract. A new system of time-dependent anharmonic and anisotropic oscillators in two space-dimensions which corresponds to unequal but related spring constants, having Hamiltonian structure, admitting quadratic invariants and accounting also for the fractional powers in the coupling terms, is found.

Keywords. Time-dependent systems; dynamical invariants; hamiltonian structure.

PACS Nos 03·40; 03·20; 02·30

From several points of view the study of a system of coupled, nonlinear second order oscillators possessing at least one invariant has become quite interesting. Ermakov [1] originally suggested a connection between the solutions of such a pair of coupled equations. In recent years, Ray and Reid [2–6] and others [7–12] have studied these systems in the context of time-dependent (TD) harmonic oscillator and with different degrees of generalizations. As a matter of fact, in their course of study Ray and Reid have evolved a method of constructing the invariant for one-dimensional TD systems known as the Ermakov method and accordingly the invariant so constructed as the Ermakov invariant. The study of these systems is not only interesting as a mathematical exercise in its own right but it may also be useful in such disciplines as laser physics [13], quantum optics [13] and fluid dynamics [23].

The most studied system is the TD harmonic oscillator, which is described by [3]

$$\ddot{x} + \omega^2(t)x = 0, \tag{1}$$

and admits an invariant

$$I = (1/2)[(x/\rho)^2 + (x\dot{\rho} - \dot{x}\rho)^2], \tag{2}$$

where $x(t)$ satisfies (1) and $\rho(t)$ satisfies the auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = 1/\rho^3. \tag{3}$$

Following Ermakov [1], Ray and Reid have studied various types of generalization of eqs (1) and (3), and obtained [4, 5] the solutions of a specific class of nonlinear equations. Besides accounting for the damping terms in (1) and (3), the most common generalization considered by them is the following:

For the forms of (1) and (3) as [2]

$$\ddot{x} + \omega^2(t)x = g(\rho/x)/(x^2\rho), \tag{4a}$$

$$\ddot{\rho} + \omega^2(t)\rho = f(x/\rho)/(x\rho^2), \tag{4b}$$

the invariant (2) takes the form

$$I = (1/2)[\phi(x/\rho) + \theta(\rho/x) + (x\dot{\rho} - \dot{x}\rho)^2] \tag{5}$$

with

$$\phi(x/\rho) = 2\int^{(x/\rho)} f(u)\mathrm{d}u; \quad \theta(\rho/x) = 2\int^{(\rho/x)} g(u)\mathrm{d}u.$$

As a special case while $x(t)$ satisfies (1), $\rho(t)$ is also found [4] to satisfy

$$\ddot{\rho} + \omega^2(t)\rho = 1/(x\rho^2)\sum_i c_i(x/\rho)^{2m_i - 1}, \quad (i = 1, 2, \ldots) \tag{6}$$

where c_i and m_i are arbitrary constants. For $i = 1$ and 2, and with $m_1 = (m-2)/2$, $m_2 = (2m-2)/2$, eq. (6) becomes

$$\ddot{\rho} + \omega^2(t)\rho = c_1 x^{m-4}\rho^{1-m} + c_2 x^{2m-4}\rho^{1-2m}. \tag{7}$$

Among other generalizations considered by Reid and Ray [5], one is in connection with the nonlinear superposition law for the solutions of higher order nonlinear equations. Such a generalization however is not of much interest in the present context. It may be mentioned that all the above generalizations of Ermakov systems are essentially for one-dimensional TD systems (in fact, for the one-dimensional TD harmonic oscillator) and $\rho(t)$ appears as an auxiliary variable needed to provide the invariant for the corresponding TD system. On the other hand, $\rho(t)$ also plays a specific role while looking for a physical interpretation of the derived invariant [12]. In any case, it does not imply the generalization of Ermakov systems to higher space-dimensions. Further, in obtaining (7) the choices of m_1 and m_2 in (6) seem rather ad hoc.

Other generalizations of Ermakov systems have recently been considered by Leach [10] and Athorne [11]. Leach finds an explanation for the nature of the Ermakov system described by

$$\ddot{x}_1 + \omega^2(t)x_1 = g_1(x_2/x_1)/x_1^3, \tag{8a}$$

$$\ddot{x}_2 + \omega^2(t)x_2 = f_1(x_2/x_1)/x_2^3, \tag{8b}$$

in terms of a symmetry algebra. In this case, a transformation of time- and space-variables, namely $T = \cot(\int\rho^{-2}\mathrm{d}t)$; $X = \rho^{-1}x_1 \csc T$; $Y = \rho^{-1}x_2 \csc T$, eliminates $\omega^2(t)$ from eqs (8) and the newly introduced variable ρ is found to satisfy an equation of the type (3). On the other hand, Althorne makes use of this symmetry algebra of Leach to analyse a Kepler–Ermakov system of the type $\ddot{\mathbf{x}} + \omega^2(t)\mathbf{x} = v(\mathbf{x})/r^3$, by setting $\omega^2(t) = 0$. Another interesting system studied in this context is that of coupled Pinney's equations [15, 16]

$$\ddot{x}_1 + \omega^2(t)x_1 = \beta x_1^{-3} - \alpha x_1 x_2^{-4}, \tag{9a}$$

$$\ddot{x}_2 + \omega^2(t)x_2 = \delta x_2^{-3} - \gamma x_2 x_1^{-4}. \tag{9b}$$

In this case, however, there remains [14], in general, a difficulty of finding a Hamiltonian structure. In the absence of such a structure the system remains of least physical interest. For example, for the system (9) the Hamiltonian structure exists

[16] only for $\alpha = 3\delta$, $\gamma = 3\beta$ and with a kinetic term of hardly any physical interest. In spite of the fact that the invariant associated with this system can be constructed explicitly, this system does not turn out [11] to be integrable even for $\omega(t) =$ constant, i.e. even in a time independent case.

It may be mentioned that the variable $x_2(t)$ in (9) can, in principle, be regarded as $\rho(t)$ of (3) or of (6), and is, in fact, a variable in the second space dimension. On the other hand, the system studied by Leach (cf. eqs (8)) no doubt reduces to the Ray and Reid-form (4) on redefining g_1 and f_1 as

$$g_1(x_2/x_1) = (x_1/x_2)g(x_1/x_2), \quad f_1(x_2/x_1) = (x_2/x_1)f(x_2/x_1)$$

but is structurally different from (4) in the sense that ρ-equation now appears as a by-product of the transformation. Also, in the work of Ray and Reid the ρ-equation (3) is a prerequisite for the existence of the invariant (2). Thus, eqs (8) and (9) although describe certain TD coupled oscillators in two dimensions (with equal spring constants along each dimension) but without any auxiliary equation. Now the question arises as to what are the Ermakov systems in two dimensions which remain integrable and also retain the Hamiltonian structure at the same time. In what follows we suggest a class of Ermakov-type systems in two dimensions which (i) deals with unequal but related spring constants, (ii) ensures, in general, a Hamiltonian structure, (iii) admits quadratic (in momenta) invariants, (iv) can involve fractional powers in the coupling terms, and (v) involves a pair of auxiliary equation in a natural manner. To the best of our knowledge, all these features in a time-dependent system have not been considered so far.

In the present work, we consider a two-dimensional system,

$$V(x_1, x_2, t) = \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 + \beta(t)x_1^m x_2^n. \tag{10}$$

It is found [17] that this system admits a quadratic invariant only for the case when the arbitrary numbers m and n satisfy the condition $m + n = -2$ and β is a constant. Due to symmetry considerations, we investigate here a system described by the Lagrangian,

$$L = (1/2)(\dot{x}_1^2 + \dot{x}_2^2) - \alpha_1(t)x_1^2 - \alpha_2(t)x_2^2 - \beta_{10}x_1^{-2-n}x_2^n - \beta_{20}x_1^n x_2^{-2-n}. \tag{11}$$

Before we proceed further a few remarks about this general form are in order: (i) Invariants corresponding to some special cases of (11) (particularly, when n is an integer, namely $n = -1, 0$ and $+1$) are derived earlier [17] using the rationalization method [17–19]. In (11) while β_{10} and β_{20} are constants, they can be time-dependent for $n = 0$. The results corresponding to fractional values of n will be discussed later as a special case. (ii) The dynamical algebraic approach [9] when extended to two-dimensional systems also leads [20] to the form (11) as a result of the closer property of the associated Lie algebra. From the rationalization of the 'potential' equation [17–19] for the system (11), the coefficient functions $a_i(x_1, x_2, t)$ and $a_{ij}(x_1, x_2, t)$ in the invariant

$$I = a_0 + a_1\dot{x}_1 + a_2\dot{x}_2 + (1/2)(a_{11}\dot{x}_1^2 + 2a_{12}\dot{x}_1\dot{x}_2 + a_{22}\dot{x}_2^2) \tag{12}$$

turn out to be

$$a_{11} = a_{22} = \psi_3(t); \quad a_{12} = 0,$$

$$a_1 = -(1/2)(\dot{\psi}_3 x_1 + c_5 x_2 - \psi_7); \quad a_2 = -(1/2)(\dot{\psi}_3 x_2 - c_5 x_1 - \psi_6),$$

where c_i are the arbitrary constants and $\psi_i \equiv \psi_i(t)$, are the arbitrary functions. A unique expression for a_0 can be obtained from the integration of the equations [17, 18]

$$\partial a_0/\partial x_1 = a_{11}(\partial V/\partial x_1) - \partial a_1/\partial t; \quad \partial a_0/\partial x_2 = a_{22}(\partial V/\partial x_2) - \partial a_2/\partial t,$$
$$\partial a_0/\partial t = a_1(\partial V/\partial x_1) + a_2(\partial V/\partial x_2)$$

as

$$a_0 = \{\alpha_1 \psi_3 + (1/4)\ddot{\psi}_3\}x_1^2 + \{\alpha_2 \psi_3 + (1/4)\ddot{\psi}_3\}x_2^2 + \beta_{10}\psi_3 x_1^{-2-n} x_2^n + + \beta_{20}\psi_3 x_1^n x_2^{-2-n}, \quad (13)$$

provided ψ_3, α_1 and α_2 satisfy the equations

$$(1/4)\dddot{\psi}_3 + 2\alpha_1 \dot{\psi}_3 + \dot{\alpha}_1 \psi_3 = 0; \quad (14a)$$
$$(1/4)\dddot{\psi}_3 + 2\alpha_2 \dot{\psi}_3 + \dot{\alpha}_2 \psi_3 = 0. \quad (14b)$$

and $c_5 = \psi_6 = \psi_7 = 0$. Equations (14a) and (14b) further provide $\psi_3(t)$ as

$$\psi_3(t) = c_3(\alpha_1 - \alpha_2)^{-1/2}, \quad (15)$$

with a constraint on the potential parameters α_1 and α_2 of the form

$$4(\dddot{\alpha}_1 - \dddot{\alpha}_2)(\alpha_1 - \alpha_2)^2 - 18(\alpha_1 - \alpha_2)(\dot{\alpha}_1 - \dot{\alpha}_2)(\ddot{\alpha}_1 - \ddot{\alpha}_2) + 15(\dot{\alpha}_1 - \dot{\alpha}_2)^3 = 32(\alpha_1 - \alpha_2)^2(\alpha_1 \dot{\alpha}_2 - \dot{\alpha}_1 \alpha_2). \quad (16)$$

Finally, the invariant (12) for the system (11) becomes

$$I = \{\alpha_1 \psi_3 + (1/4)\ddot{\psi}_3\}x_1^2 + \{\alpha_2 \psi_3 + (1/4)\ddot{\psi}_3\}x_2^2 + \beta_{10}\psi_3 x_1^{-2-n} x_2^n + + \beta_{20}\psi_3 x_1^n x_2^{-2-n} - (1/2)\dot{\psi}_3(x_1 \dot{x}_1 + x_2 \dot{x}_2) + (1/2)\psi_3(\dot{x}_1^2 + \dot{x}_2^2), \quad (17)$$

which, in fact, is in conformity with $\mathrm{d}I/\mathrm{d}t = \partial I/\partial t + [I, H]_{PB} = 0$.

Several remarks about the form (11) and the corresponding invariant (17) are in order:

(1) For $n = -1$ and $+1$, the form (17) not only reduces to the results obtained earlier [17] but also provides the invariants for a class of systems characterized by the arbitrary number n including its fractional values. For example, for $n = -3/2$ in (11) the invariant (17) takes the form

$$I = \{\alpha_1 \psi_3 + (1/4)\ddot{\psi}_3\}x_1^2 + \{\alpha_2 \psi_3 + (1/4)\ddot{\psi}_3\}x_2^2 + \beta_{10}\psi_3 x_1^{-1/2} x_2^{-3/2} + \beta_{20}\psi_3 x_1^{-3/2} x_2^{-1/2} - (1/2)\dot{\psi}_3(x_1 \dot{x}_1 + x_2 \dot{x}_2) + (1/2)\psi_3(\dot{x}_1^2 + \dot{x}_2^2) \quad (18)$$

with ψ_3 again given by (15) and α_1 and α_2 again related through (16). In this case, however, the invariant I is constrained by the values of x_1 and x_2, particularly when they are negative. In fact, one should consider $|I|$ here for negative values of x_1 or of x_2.

(2) The pair of coupled equations of motion corresponding to the system (11) can be written as

$$\ddot{x}_1 + 2\alpha_1(t)x_1 = (n+2)\beta_{10} x_1^{-3-n} x_2^n - n\beta_{20} x_1^{n-1} x_2^{-2-n}, \quad (19a)$$

$$\ddot{x}_2 + 2\alpha_2(t)x_2 = -n\beta_{10}x_1^{-2-n}x_2^{n-1} + (n+2)\beta_{20}x_1^{n}x_2^{-3-n}. \tag{19b}$$

It is interesting to note that eqs (19a) and (19b) while providing a physically acceptable Hamiltonian structure also offer a Pinney type system [11] for $n = 1$. Further, these equations alongwith eqs (14) not only suggest a new class of time-dependent systems for any finite n but can also be regarded as a generalization of eqs (1) and (7) to higher space-dimensions. In this latter case $\psi_3(t)$, satisfying a pair of third-order equations (of, eq. (14)), can be considered as playing the role of an auxiliary variable (just like $\rho(t)$ in one dimension).

(3) In order to get (7) from (6) the choice of m_1 and m_2 in terms of m seems rather ad hoc in the method of Ray and Reid; whereas here in the two-dimensional case there exists [17] a rationale for such a choice which, in fact, reflects the characteristic of the dynamical system (10).

(4) The invariant (17) is also obtained using other [20] methods and it can be expressed as

$$I = \psi_3(t)H(t) + (1/4)\mathrm{d}[\dot{\psi}_3(x_1^2 + x_2^2)]/\mathrm{d}t - \dot{\psi}_3(x_1\dot{x}_1 + x_2\dot{x}_2) \tag{20}$$

where $H(t)$ is the Hamiltonian corresponding to the system (11). Equation (20) implies that for $\dot{\psi}_3 = 0$, the invariant I reduces to $H(t)$ and α_1 and α_2 become time-independent (cf. eq. (14)).

To summarize, we mention that there exists a new class of time-dependent systems in two dimensions which not only possesses Hamiltonian structure and admits quadratic invariants but can also account for the fractional powers in the coupling terms. Further, there exists a rationale for the powers of the coupling terms in these systems. While all these features could be important in characterizing the new Ermakov-type systems, they are somehow not exhibited by some of the known systems like Pinney's equations. For the complete integrability of these systems, however, there is a need for one more invariant to exist [21]. Such studies with reference to higher order [22] (in momenta) invariants are in progress.

Acknowledgements

The author wishes to record his sincere thanks to Dr D Parashar for several discussions and for critical reading of the manuscript. Several useful suggestions by the referee are gratefully acknowledged. Financial grant from UGC, New Delhi in the form of Research Scientist Scheme is also acknowledged.

References

[1] V Ermakov, *Univ. Izv. Kiev series III* **9**, 1 (1880)
[2] J R Ray and J L Reid, *Phys. Lett.* **A71** 317 (1979)
[3] J R Ray and J L Reid, *Phys. Lett.* **A74**, 23 (1979)
[4] J R Ray and J L Reid, *J. Math. Phys.* **20**, 2054 (1979)
[5] J L Reid and J R Ray, *J. Math. Phys.* **23**, 503 (1982)
[6] J L Reid and J R Ray, *J. Phys.* **A15**, 2751 (1982)
[7] H Lewis and W Riesenfeld, *J. Math. Phys.* **10**, 1458 (1969)
[8] M Lutzky, *Phys. Lett.* **A71**, 317 (1979); **A78** 301 (1980)
[9] R S Kaushal and H J Korsch, *J. Math. Phys.* **22** 1904 (1981)
[10] P G L Leach, *Phys. Lett.* **A158**, 102 (1991)

[11] C Athorne, *Phys. Lett.* **A159**, 375 (1991)
[12] C J Eliezer and A Gray, *SIAM J. Appl. Math.* **30**, 463 (1976)
[13] J M Cervero and J D Lejarreta, *Quant. Opt.* **2**, 333 (1990)
[14] J M Cervero and J D Lejarreta, *Phys. Lett.* **A156**, 201 (1991)
[15] E Pinney, *Proc. Am. Math. Soc.* **1**, 681 (1950)
[16] J Douglas, *Trans. Am. Math. Soc.* **50**, 71 (1941)
[17] R S Kaushal, D Parashar and S C Mishra, *Second order invariants for two dimensional classical dynamical systems* to be published in *Fortschritte der Physik* **42** (1994) #5 (in press)
[18] R S Kaushal, S C Mishra and K C Tripathy, *Phys. Lett.* **A102**, 7 (1984)
S C Mishra, R S Kaushal and K C Tripathy, *J. Math. Phys.* **25**, 2217 (1984)
[19] S C Mishra *Some aspects of two dimensional classical integrable systems*, Ph D Thesis (Univ. of Delhi, 1985), unpublished
[20] R S Kaushal and S C Mishra, *J. Math. Phys.* **34**, 5843 (1993)
[21] E T Whittaker, *Analytical dynamics* (University Press, Cambridge, 1927)
[22] R S Kaushal, *Pramana – J. Phys.* **24**, 663 (1985)
[23] For other applications of these systems, see, for example, the unpublished recent work cited in [11]