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On phases and length of curves in a cyclic quantum evolution

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Abstract. The concept of a curve traced by a state vector in the Hilbert space is introduced into the general context of quantum evolutions and its length defined. Three important curves are identified and their relation to the dynamical phase, the geometric phase and the total phase are studied. These phases are reformulated in terms of the dynamical curve, the geometric curve and the natural curve. For any arbitrary cyclic evolution of a quantum system, it is shown that the dynamical phase, the geometric phase and their sums and/or differences can be expressed as the integral of the contracted length of some suitably-defined curves. With this, the phases of the quantum mechanical wave function attain new meaning. Also, new inequalities concerning the phases are presented.

Keywords. Geometric phase; distance function; length of the curves.

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1. Introduction

The phase change of the quantum mechanical wave function is of vital importance as it gives information about the dynamical changes in the system as well as the geometry of the path of the evolution. This paper aims at giving a new meaning towards all phase changes in general. When we are dealing with the cyclic evolution of a physical system, the possible natural questions that concern us are: (i) what is the phase factor that is associated with the final state (ii) how much distance the quantum state has travelled during the period of evolution in the projective Hilbert space $\mathscr{P}$ as measured by the Fubini–Study metric and (iii) what is the total length of the curve traced by the state vector (or other curves traced by the phase-transformed vectors)? The answer to (i) is well-known: the total phase factor contains two parts. One is the usual dynamical phase that gives the information about the duration of the evolution of the system and the second phase factor is the geometric phase (Berry phase [1, 2]) that gives the information about the geometry of the path of the evolution in $\mathscr{P}$ [3, 4]. The answer to the question (ii), though not widely-studied, has drawn some attention in connection with the introduction of geometric structures into quantum mechanics [5–14] and in relating the geometric phase to the geometric distance function [11–13]. The total distance travelled by the quantum states along a given curve $\hat{C}$ in $\mathscr{P}$, as measured by the Fubini–Study metric [7–13] is the time integral of the uncertainty in the energy of the system. Anandan and Aharonov [7] have remarked that it is also a geometric quantity analogous to the geometric phase, in the sense that it does not depend on the particular Hamiltonian used to move the quantum system along a given curve $\hat{C}$ in $\mathscr{P}$. In fact these are not just analogies but are related quantities. Recently, the author [11] has shown that the geometric phase could be derived from metric considerations. It appears as a result of the principle of minimization of the speed of transportation in $\mathscr{P}$. However the question (iii) has not drawn much attention. Recently we [11, 13] have asked such a question in connection with the geometric phase. For any cyclic evolution, when we say the geometric phase depends only on the closed curve $\hat{C}$ in $\mathscr{P}$ it is not transparent in what way the phase depends on the curve (unless we define the curve and its length).

In this paper, we introduce the concept of a curve traced by a normalized vector in the Hilbert space $\mathscr{H}$ and give a general definition of the length of a curve during an arbitrary cyclic quantum evolution. Next, we elucidate the properties of the length of a curve. In the sequel, we define three important curves and their lengths. We study their relation to the dynamical phase and the geometric phase for cyclic evolutions of the quantum system. Any phase (we mean dynamical, geometric and/or their sums and differences) can be expressed as an integral of the contracted length of (some suitably-defined) curves. With these curves, the phases of the quantum mechanical wave function attain new meanings. For all cyclic evolutions of the quantum system we show that the value of the phases cannot be arbitrarily large, but are limited by the total length of their corresponding curves. Furthermore each curve is characterized by a number called energy associated with the curve. We point out the importance of the energy of the curve by carrying out a variational calculation leading to the equation of geodesic. We obtain some new inequalities among phases, lengths and energies for various kinds of curves in the Hilbert space. Lastly, we study a simple example to illustrate the concepts that are introduced here.

2. Concept of various curves and their lengths

To be concrete we start this section by defining a curve traced by any arbitrary vector in the Hilbert space $\mathscr{H}$ and its length. Let $|\chi(t)\rangle$ be a unit-norm vector belonging to $\mathscr{H}$. When $|\chi(t)\rangle$ evolves in time it traces a curve in $\mathscr{H}$, i.e. $t \to |\chi(t)\rangle$, $0 \leqslant t \leqslant T$ is a curve on the unit ball $\mathbb{B}$ of the Hilbert space $\mathscr{H}$. Since $\mathscr{H}$ is Riemannian this curve has a length. This curve projects to either a closed or an open curve in the projective Hilbert space $\mathscr{P} = \mathscr{B}/U(1) = \mathscr{H} - \{0\}/\mathbb{C}^*$ depending on the type of evolution, named cyclic or non-cyclic. The projected curve $\hat{C} = \pi(C)$ also has a length, given by the Fubini–Study metric on $\mathscr{P}$. Thus if $|\chi(t)\rangle$ is a quantum state then we say that the state of the system at any instant of time is represented by a point and the evolution of the state is by a curve in $\mathscr{P}$. Since the inner product in $\mathscr{H}$ induces a metric in $\mathscr{P}$, we can define the length of the curve precisely.

Definition Let $t \in [0, T] \subset \mathbb{R}$ and let $C: t \to \chi(t)$ be a piecewise C^1 curve on the unit ball $\mathscr{B} \subset \mathscr{H}$. Its length is defined to be

$$l(\chi(t))|_0^T = \int_0^T \langle \dot{\chi}(t)|\dot{\chi}(t)\rangle^{1/2} \mathrm{d}t, \tag{1}$$

where $|\dot{\chi}(t)\rangle$ is the velocity vector in the Hilbert space $\mathscr{H}$ of the curve $|\chi(t)\rangle$ at point t along the path of evolution of the state vector.

Some of the properties of the length of a curve can be spelled out. First of all the integral (1) exists since the integrand is continuous due to the map: $t \to \|\dot{\chi}\|$ being continuous. The length of a broken C-curve is defined as the (finite) sum of the length of its C-pieces. This means that the broken curve $t \to \chi(t)$ is a broken C^∞-curve in

$\mathscr{H}$ and it is a continuous mapping $C:[0,T]\to\mathscr{H}$ together with a subdivision $0\leqslant t_1<t_2<\cdots<t_n\leqslant T$ on whose closed subinterval $t\to\chi(t)$ is a C^∞-curve. The length of the curve has an important property of reparametrization invariance. Given a smooth curve C and two fixed points $\chi(0)$ and $\chi(T)$ corresponding to the parameter values $t=0$ and $t=T$, if we change the parameter t by τ that is an arbitrary smooth transformation and a smooth monotonically increasing function of time (i.e. $dt/d\tau>0$), then the length of the curve $l(\chi)|_0^T$ remains unchanged. The curve C can be viewed as a continuous set of points in $\mathscr{H}$ along with a parametrization. Thus length is a geometric property of the whole curve C in $\mathscr{H}$, defined by the equivalence classes of parametrized paths. Hence the length of the curve defined from the inner product of the tangent vectors is an important concept in studying the geometry of quantum evolution.

It is now convenient to define the infinitesimal length of the curve during an infinitesimal time, dt, for an arbitrary time evolution of the quantum system as follows:

$$dl(\chi(t))=\langle\dot\chi(t)|\dot\chi(t)\rangle^{1/2}dt. \tag{2}$$

This will help us in studying the infinitesimal properties of a curve on a manifold of quantum states. If the parameter t is such that $\langle\dot\chi|\dot\chi\rangle^{1/2}$ is constant along C, then we say the length of the curve is parametrically proportional to the arc length. The quantity $u_{\mathscr{H}}(\chi)=dl(\chi)/dt$ is called the magnitude of the rate of change of arc length of the curve C in $\mathscr{H}$. This can be seen by changing the parameter from t to τ, the curve $\tau\to|\chi(\tau)\rangle$ will then be the locus of the same points in $\mathscr{H}$ as for the curve $t\to|\chi(t)\rangle$, nevertheless, the state traverses along the curve at a different rate because $u_{\mathscr{H}}(\chi)$ is not reparmeterization invariant.

We now introduce various curves such as the dynamical curve, the geometric curve and the natural curve and define their lengths. These lengths share all the properties elucidated in the general discussion.

Dynamical curve: Let $|\psi(t)\rangle$ be the quantum state that evolves according to the Schrödinger equation $i\hbar|\dot\psi(t)\rangle=H(t)|\psi(t)\rangle$. The curve $t\to|\psi(t)\rangle$ traced by the vector $|\psi(t)\rangle$ is called as the dynamical curve C_d. This satisfies $|\psi(T)\rangle=\exp(i\phi)|\psi(0)\rangle$, $\phi\in\mathbb{R}$ (phase ϕ is called as the total phase). It begins and ends on the same ray but at different points.

The length of the dynamical curve C_d can be defined through (1) by taking $|\chi(t)\rangle$ to be $|\psi(t)\rangle$. Since it depends on the Hamiltonian of the system (in view of $|\psi(t)\rangle$ satisfies the Schrödinger equation corresponding to a given Hamiltonian) and consequently on the detailed dynamics, hence the name dynamical curve.

Geometric curve: The curve $t\to|\tilde\psi(t)\rangle$ traced by the single valued vector $|\tilde\psi(t)\rangle$ is called as the geometric curve C_g. It is a phase-transformed solution curve of $|\psi(t)\rangle$ (because $|\tilde\psi(t)\rangle=\exp(-if(t))|\psi(t)\rangle$ with $f(T)-f(0)=\phi$) such that $|\tilde\psi(T)\rangle=|\tilde\psi(0)\rangle$. It starts and ends on the same ray at same point, i.e. C_g is a closed curve in $\mathscr{H}$ with $\pi(C_g)=\pi(C_d)=\hat{C}$.

The length of the geometric curve can be defined through (1) by taking $|\chi(t)$ to be $|\tilde\psi(t)\rangle$. It is independent of the particular Hamiltonian used to transport the quantum system. This is true to the extent that an addition of a scalar E to the Hamiltonian, ensues a different total phase factor to the evolving state keeping the single valued state unchanged. That is why we call the curve traced by the single-valued vector

$|\tilde{\psi}(t)\rangle$ as a geometric curve. Since the single valued vector only depends on the image of the evolution of $|\psi(t)\rangle$ on the projective Hilbert space $\mathscr{P}$, it further consolidates the geometric nature of the curve C_g.

Natural curve: The curve $t \to |\bar{\psi}(t)\rangle$ traced by the parallel-transported vector $|\bar{\psi}(t)\rangle$ is called as the natural curve C_n. This is also a phase-transformed solution curve of $|\psi(t)\rangle$ (because $|\bar{\psi}(t)\rangle = \exp(i/\hbar \int_0^t \langle\psi(t')|H(t')|(t')\rangle dt')|\psi(t)\rangle$, with the parallel transport rule $\langle\bar{\psi}(t)|\dot{\bar{\psi}}(t)\rangle = 0$) such that $|\bar{\psi}(T)\rangle = \exp(i\beta)|\bar{\psi}(0)\rangle$, where β is the geometric phase acquired by the system during a cyclic evolution. This curve starts and ends on the same ray but at different points.

The length of the natural curve can be defined in the same way by taking $|\chi(t)\rangle$ to be $|\bar{\psi}(t)$ in the definition (1). Like the length of the geometric curve it is also $H(t)$- and t-invariant quantity.

For further clarification, we can see that the length of the natural curve is exactly equal to the total distance travelled by the state vector $|\psi(t)\rangle$ along a given curve $\hat{C}$ in the projective Hilbert space $\mathscr{P}$. More precisely, the projective Hilbert space $\mathscr{P}$ admits a natural metric structure and that the distance as measured by the metric topology coincides with the length of the natural curve. Thus $l(\bar{\psi})|_0^T = D$, where $D = \int_0^T \Delta E(t) dt/\hbar$ and $\Delta E(t) = [\langle\psi(t)|H^2(t)|\psi(t)\rangle - \langle\psi(t)|H(t)|\psi(t)\rangle^2]^{1/2}$. Therefore, the total length of the natural curve is equal to the time integral of the uncertainty of the energy during the period of the evolution.

3. Concept of energy of the curve

After studying different curves traced by the equivalence classes of state vectors, we now introduce the energy of the various curves. With this, any arbitrary evolution can be classified by the length and energy of the curve [15].

Proposition To each curve, there is associated an energy, which is distinct from the length of the curve.

The number defined through

$$E(\chi(t))|_0^T = \hbar \int_0^T \langle\dot{\chi}(t)|\dot{\chi}(t)\rangle dt \tag{3}$$

is called the energy associated with the curve $t \to |\chi(t)\rangle$. If we allow $|\chi(t)\rangle$ to be $|\psi(t)\rangle$, $|\tilde{\psi}(t)\rangle$ and $|\bar{\psi}(t)\rangle$ we will get the quantum mechanical energies of the dynamical, geometric and natural curves respectively. We denote these energies by $E(\psi)$, $E(\tilde{\psi})$ and $E(\bar{\psi})$ respectively.

To avoid confusion among the readers we would like to remark that the energy of the curve $E(\chi)$ has nothing to do with the true quantum mechanical energy of the system, in general. The energy of a curve is a property that is attributed to a particular curve during an evolution of a quantum system. The true quantum mechanical energy E_s of a system is given by the expectation value of the hamiltonian in the state $|\psi(t)\rangle$ and is a property of the system at a given instant of time. So these two energies are different altogether. However, the energy of the dynamical curve and the true quantum mechanical energy of the system obey a simple inequality. Since the infinitesimal change in the energy of the dynamical curve is given by $dE(\psi) = \hbar\, \langle\dot{\psi}(t)|\dot{\psi}(t)\rangle dt$, we have $E_s^2 \leqslant \hbar dE(\psi)/dt$. Here, we have used the fact that the variance in the energy of the system is a non-negative quantity. Thus the (squared) energy of the system is limited by the rate of change of energy of the dynamical curve.

Next, we make a comparison between the length and the energy of the curve. We state a theorem which elicits the important fact that the length and the energy of the curve are distinct objects in the Hilbert space $\mathscr{H}$.

Theorem 1. The (squared) length of the curve during a cyclic evolution of the quantum system is limited by the energy of the curve, i.e. the length and the associated energy of the curve are related by the inequality

$$l^2(\chi(t))|_0^T \leqslant E(\chi(t))|_0^T (T/\hbar). \tag{4}$$

The equality holds good for a curve which is parametrically proportional to the arc length of the curve, i.e., when $\langle \dot{\chi}(t)|\dot{\chi}(t)\rangle$ is constant. We have three such inequalities for each of the curves mentioned above.

4. Variation of the energy of the curve and geodesic equation

In this section we outline as to how the concept of energy of the curves can be used to define geodesics by studying their extremal properties under smooth variations. Among all other curves and energies, the natural curve and its associated energy deserve a separate status in studying the geometry of quantum evolution. On a Riemannian manifold of quantum states a geodesic can be defined as a curve traced by the parallel transported vector $|\bar{\psi}(t)\rangle$ along which the translation preserves the velocity vector of the curve C_n. Alternatively, geodesics in $\mathscr{P}$ can be defined as those curves for which the energy of the natural curve is stationary. We now present a variational argument to arrive at a geodesic equation.

Using the definition of the energy of the natural curve we write the energy of the varied curve as

$$\delta E(\bar{\psi})|_0^T = \hbar \int_0^T \mathrm{d}t \left[\mathrm{Re}\left(\left\langle \frac{\partial \dot{\psi}}{\partial u} \middle| \dot{\psi} \right\rangle - \left\langle \frac{\partial \dot{\psi}}{\partial u} \middle| \psi \right\rangle \langle \psi|\dot{\psi}\rangle - \langle \dot{\psi}|\psi\rangle \left\langle \frac{\partial \psi}{\partial u} \middle| \dot{\psi} \right\rangle \right) \right] \mathrm{d}u. \tag{5}$$

On using

$$\left\langle \frac{\partial \dot{\psi}}{\partial u} \middle| \frac{\delta \psi}{\mathrm{d}t} \right\rangle = \left\langle \frac{\partial \dot{\psi}}{\partial u} \right| (|\dot{\psi}\rangle - \langle \psi|\dot{\psi}\rangle|\psi\rangle),$$

we have

$$\delta E(\bar{\psi})|_0^T = \hbar \int_0^T \mathrm{d}t \left[\mathrm{Re}\left(\left\langle \frac{\partial \dot{\psi}}{\partial u} \middle| \frac{\delta \psi}{\mathrm{d}t} \right\rangle - \left\langle \frac{\partial \psi}{\partial u} \middle| \dot{\psi} \right\rangle \langle \dot{\psi}|\psi\rangle \right) \mathrm{d}u \right], \tag{6}$$

where

$$\left| \frac{\delta \psi}{\mathrm{d}t} \right\rangle = |\dot{\psi}\rangle - \langle \psi|\dot{\psi}\rangle|\psi\rangle.$$

Notice that $\langle \psi|\dot{\psi}\rangle$ and $\left\langle \psi \middle| \frac{\partial \psi}{\partial u} \right\rangle$ are purely imaginary. Facilitating the usage of the identity $\mathrm{Re}\left[\mathrm{d}u \left\langle \frac{\partial \psi}{\partial u} \middle| \frac{\delta \psi}{\mathrm{d}t} \right\rangle \langle \psi|\dot{\psi}\rangle \right] = -\mathrm{Re}\left[\mathrm{d}u \left\langle \frac{\partial \psi}{\partial u} \middle| \dot{\psi} \right\rangle \langle \dot{\psi}|\psi\rangle \right]$, we have

$$\delta E(\bar{\psi})|_0^T = \hbar \int_0^T \mathrm{d}t \left[\mathrm{Re}\left(\left\langle \frac{\partial \dot{\psi}}{\partial u} \middle| \frac{\delta \psi}{\mathrm{d}t} \right\rangle + \left\langle \frac{\partial \psi}{\partial u} \middle| \frac{\delta \psi}{\mathrm{d}t} \right\rangle \langle \psi | \dot{\psi} \rangle \right) \mathrm{d}u \right]. \tag{7}$$

On integrating the first term by parts and allowing the variation of $|\psi(u,t)\rangle$ at end points to vanish, we obtain

$$\delta E(\bar{\psi})|_0^T = \hbar \int_0^T \mathrm{d}t \left[\mathrm{Re}\left(\left\langle \frac{\partial \psi}{\partial u} \middle| \left(-\frac{\mathrm{d}}{\mathrm{d}t} + \langle \psi | \dot{\psi} \rangle \right) \middle| \frac{\delta \psi}{\mathrm{d}t} \right\rangle \right) \mathrm{d}u \right]. \tag{8}$$

The equation for the geodesic comes naturally, if one observes that $\left\langle \psi \middle| \frac{\partial \psi}{\partial u} \right\rangle$ is purely imaginary and the integrand vanishes for arbitrary variations. This amounts to saying that [16]

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \langle \psi | \dot{\psi} \rangle \right) \left| \frac{\delta \psi}{\mathrm{d}t} \right\rangle = g(t) |\psi(t)\rangle, \tag{9}$$

for any real, smooth function $g(t)$. This is the geodesic equation of motion for the system. Evolutions of the quantum system for which the energy of the natural curve is stationary satisfies (9). In terms of the parallel-transported vector the geodesic equation is simple and is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \left| \bar{\psi}(t) \right\rangle + v_{\mathscr{H}}^2(t) |\bar{\psi}(t)\rangle = 0, \tag{10}$$

where the function $g(t)$ is eliminated by using the parallel-transport rule $\langle \bar{\psi} | \dot{\bar{\psi}} \rangle = 0$, and is found to be $v_{\mathscr{H}}$, where $v_{\mathscr{H}} = \mathrm{d}l(\bar{\psi})/\mathrm{d}t$.

Since the geodesics are special paths, it would be interesting to know the phase information of a quantum system when it passes through a geodesic. Especially, we would like to know what is the geometric phase acquired by a quantum system when it passes through a geodesic. The following theorem answers this question.

Theorem 2. For a time independent Hamiltonian, if the system passes through geodesic (meaning, (10) is satisfied by the parallel-transported vector) and further, the tangent vector is proportional to the vector itself at time $t = 0$, i.e. $|\dot{\bar{\psi}}(0)\rangle = i v_{\mathscr{H}} |\bar{\psi}(0)\rangle$, then the geometric phase acquired by the system in one cycle is just equal to the total length of the natural curve during the evolution.

Proof. The general solution to (10) is

$$|\bar{\psi}(t)\rangle = (\cos v_{\mathscr{H}} t) |\bar{\psi}(0)\rangle + (\sin v_{\mathscr{H}} t)/v_{\mathscr{H}} |\dot{\bar{\psi}}(0)\rangle. \tag{11}$$

Imposing the condition $|\dot{\bar{\psi}}(0)\rangle = i v_{\mathscr{H}} |\bar{\psi}(0)\rangle$ we can write $|\bar{\psi}(t)\rangle = \exp(i v_{\mathscr{H}} t) |\bar{\psi}(0)\rangle$. In one cycle, $|\bar{\psi}(T)\rangle$ is related to $|\bar{\psi}(0)\rangle$ through $|\bar{\psi}(T)\rangle = \exp(i v_{\mathscr{H}} T) |\bar{\psi}(0)\rangle = \exp(i l(\bar{\psi})|_0^T) |\bar{\psi}(0)\rangle$. But $|\bar{\psi}(T)\rangle$ is again equal to $\exp(i\beta(\mathrm{C})) |\bar{\psi}(0)\rangle$ where $\exp(i\beta(\mathrm{C}))$ is the holonomy transformation in parallel-transporting the vector $|\bar{\psi}(t)\rangle$ arround a closed curve in the projective Hilbert space $\mathscr{P}$ and $\beta(\mathrm{C})$ is the geometric phase acquired by the system during a cyclic evolution, hence the proof.

5. Relating phases and length of the curves

In this section we relate the phases and length of the curves during any cyclic quantum evolution. Before doing so, we state the following theorem which is of paramount importance in proving the central result of the paper.

Theorem 3. The length of the curves traced by any equivalence classes of states is greater than the total length of the natural curve during an arbitrary cyclic evolution of the quantum system. (Here, equivalence classes we mean vectors differing only in phases. The equivalence relation is $|\psi\rangle \sim |\psi'\rangle$ if $|\psi'\rangle = c|\psi\rangle$ where $0 \neq c \in \mathbb{C}^*$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$ is a multiplicative group of non-zero complex numbers.)

This is reminiscent of a theorem that is known in differential geometry [17]. From theorem 3, it immediately follows that the total length of the dynamical and geometric curves is greater than the total length of the natural curve during an arbitrary cyclic evolution. Also, it is easy to show that the total energy of the dynamical and geometric curves is greater than the total energy of the natural curve.

Theorem 4. For an arbitrary cyclic evolution of the quantum system, (if the frequencies (change of phases over unit time interval) do not change sign during the evolution) then the dynamical phase, the geometric phase, their sums (called total phase) and their differences can be expressed as an integral of the contracted length of some (suitably-defined) curves.

Proof. For a cyclic quantum evolution, the dynamical phase is the time integral of the expectation value of the Hamiltonian $H(t)$, i.e. $\delta = -1/\hbar \int_0^T \langle\psi|H(t)|\psi\rangle \mathrm{d}t$. This can be expressed as

$$\delta = \int_0^T (1 - v_{\mathscr{H}}^2(\bar{\psi})/u_{\mathscr{H}}^2(\psi))^{1/2} \mathrm{d}l(\psi). \tag{12}$$

From the above theorem, we know that $u_{\mathscr{H}}(\psi) > v_{\mathscr{H}}(\bar{\psi})$, where $u_{\mathscr{H}}(\psi)$ is the rate of change of arc length of the dynamical curve and $v_{\mathscr{H}}(\bar{\psi})$ is the rate of change of arc length of the natural curve (also called as the speed of transportation of the state vector in the projective Hilbert space $\mathscr{P}$). We call the quantity $[(1 - v_{\mathscr{H}}^2(\bar{\psi})/u_{\mathscr{H}}^2(\psi)]^{1/2}$ as the contraction factor (CF) for the dynamical curve. Hence the magnitude of the dynamical phase is the integral of the contracted length of the dynamical curve.

The geometric phase for a cyclic quantum evolution is given by $\beta(\mathrm{C}) = i\int_0^T \langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle \mathrm{d}t$, which can be expressed as

$$\beta(c) = \int_0^T [1 - v_{\mathscr{H}}^2(\bar{\psi})/u_{\mathscr{H}}^2(\tilde{\psi})]^{1/2} \mathrm{d}l(\tilde{\psi}). \tag{13}$$

Since $u_{\mathscr{H}}(\tilde{\psi}) > v_{\mathscr{H}}(\bar{\psi})$, where $u_{\mathscr{H}}(\tilde{\psi})$ is the rate of change of arc length of the geometric curve, we call the quantity $[(1 - v_{\mathscr{H}}^2(\bar{\psi})/u_{\mathscr{H}}^2(\tilde{\psi}))]^{1/2}$ as the CF for the geometric curve. Hence the magnitude of the geometric phase is the integral of the contracted length of the geometric curve.

The total phase (difference phase) is just $\phi_+ = \phi = \delta + \beta(\phi_- = \delta - \beta)$. To express this in terms of some suitably defined length of the curve let us define a vector $|\psi_+(t)\rangle$

($|\psi_-(t)\rangle$) as follows:

$$|\psi_\pm(t)\rangle = \exp\left(\pm i\int_0^t i\langle\tilde{\psi}(t')|\dot{\tilde{\psi}}(t')\rangle \mathrm{d}t'\right)|\psi(t)\rangle. \tag{14}$$

One can see that the phases ϕ_+ and ϕ_- can be expressed as

$$\phi_\pm = \mp i\int_0^T \langle\psi_\pm(t)|\dot{\psi}_\pm(t)\rangle \mathrm{d}t. \tag{15}$$

Correspondingly, the total length of the curve traced by the vector $|\psi_\pm(t)\rangle$ is given by

$$l(\psi_\pm(t))|_0^T = \int_0^T \langle\dot{\psi}_\pm(t)|\dot{\psi}_\pm(t)\rangle^{1/2} \mathrm{d}t. \tag{16}$$

On evaluating the infinitesimal length of the curve $\mathrm{d}l(\psi_\pm)$ we get

$$\mathrm{d}l^2(\psi_\pm) - \mathrm{d}l^2(\tilde{\psi}) = (\mp i\langle\psi_\pm(t)|\dot{\psi}_\pm(t)\rangle \mathrm{d}t)^2. \tag{17}$$

Therefore $\phi_\pm$ is given by

$$\phi_\pm = \int_0^T [1 - v_{\mathscr{H}}^2(\tilde{\psi})/u_{\mathscr{H}}^2(\psi_\pm)]^{1/2} \mathrm{d}l(\psi_\pm), \tag{18}$$

which is nothing but integral of the contracted length of the curve $t \to |\psi_\pm(t)\rangle$ during the period of evolution. This proves theorem 4.

This is one of the main results of our paper. It provides expressions for any given phases in terms of the integral of the contracted length of some (suitably-defined) curves. Moreover, it establishes an equivalence between the lengths and phases in quantum theory. It can be seen as follows. If the rate of change of the length of the natural curve $v_{\mathscr{H}}(\tilde{\psi})$ is much smaller than the rate of change of the length of the other curves, then to the lowest order in $(v_{\mathscr{H}}^2/u_{\mathscr{H}}^2)$ the phases are equal to the total length of their respective curves. Thus a new interpretation of the phases is that they are all of some form or the other of the length of the curves. A necessary and sufficient condition for acquiring a phase is that the corresponding CF should not vanish during a cyclic evolution of the quantum system. Since the CF varies from zero to unity, it sets a limit on the respective phases. During such evolutions of the quantum systems the maximum attainable value of the phases are therefore equal to the total length of the respective curves (of course mod 2π). For constant CF's the phases are proportional to the total length of the corresponding curves. This statement is parallel to the well-known fact that the dynamical phase gives the duration of the period of evolution (since it is proportional to the time period for a constant energy). Here it is transparently seen that the phases also give information about the total length of the curves.

Reformulation of phases in terms of the length of the curves leads us to present some inequalities among the phases and the energies.

$$\delta \leqslant [E(\psi)T/\hbar]^{1/2}, \quad \beta \leqslant [E(\tilde{\psi})T/\hbar]^{1/2} \text{ and } \phi_\pm \leqslant [E(\psi_\pm)T/\hbar]^{1/2}. \tag{19}$$

This is true, because the phases are limited by the length of the respective curves and the lengths are limited by their energies. The physical relevance of these inequalities are as follows. In an experiment if one measures the phas change of a wave function during a cyclic quantum evolution and obtains some value then he may not be sure whether the phase is of dynamical or geometric origin. However, if he compares the measured value of the phase with the (calculated) length or energy of the corresponding curve and finds that the value of the phase satisfies any one of the inequality then he will be sure about the nature of the phase. Hence, these inequalities can be used to discern the phases in a cyclic quantum evolution.

Apart from all this, we now raise the fundamental question on the origin of the phases in general. Indeed, our result holds the clue providing a novel way towards an answer. At each instant of time, t, the length of the curves is greater than the "distance" or the length of the natural curve. Because of these fundamental inequalities among these curves, the phases appear in the final state of the system. Thus, during a cyclic quantum evolution, we may regard the excess length of curves over the length of the natural curve go on accumulating, so that their integrated squared difference finally appear as the phases associated with the final state vector. This shows that during a cyclic evolution, the nature of the phase that appears depends squarely on the particular length of the curve. An important question that we wish to answer is that whether the concept of length of a curve provides the means of realising the accumulation of a surplus geometric phase over and above the dynamical phase. This is, in fact, achieved by adjusting the Hamiltonian of the system for the following reason. Since length of the natural curve $l(\bar{\psi})$ is independent of the Hamiltonian H, by changing H we can only change the length of the dynamical curve. Therefore by varying the Hamiltonian we can allow the length of the dynamical curve to be as close as to the length of the natural curve, and thereby, making the dynamical phase vanishingly small. In the process of adjusting the Hamiltonian the length of the geometric curve remains unchanged and hence the geometric phase, too. So whatever geometric phase the system will acquire upon a cyclic excursion, can be kept intact with a very small contribution from the dynamical phase. Making the dynamical phase vanishingly small amounts to saying that the state vector of the quantum system undergoes parallel-transport arround a closed curve approximately. What kind of Hamiltonian will lead to such a situation is work of a separate paper.

6. An example

The above ideas can be clarified in the following simple example. We consider a two-level quantum system which has an analogous description of a spin-1/2 particle. The quantum system is a single, two level atom driven by an intense, classical laser field, for which the Hamiltonian in the rotating wave approximation [18] can be written as $H = 1/2\hbar\Delta\sigma_3 + \hbar\lambda[\sigma_+ \exp(-i\varphi) + \sigma_- \exp(i\varphi)]$ where σ_3 is the Pauli spin matrix, Δ is the laser frequency, λ is the coupling strength and φ is the constant phase of laser field; σ_+ and σ_- being the Pauli raising and lowering operators. In this case, the Hilbert space is spanned by the basis vectors $|e\rangle$ and $|g\rangle$; therefore $\mathscr{H} = \mathrm{C}^2$ and the projective space $\mathscr{P} = \mathscr{P}_1(\mathrm{C})$ is a sphere. If the initial state $|\psi(0)\rangle$ is chosen to be $|e\rangle$, where

$$|e\rangle = \sin\theta/2 e^{i\varphi}|+\rangle + \cos\theta/2|-\rangle, \tag{20}$$

then, at any other time the state vector $|\psi(t)\rangle$ can be written as

$$|\psi(t)\rangle = \sin\theta/2 e^{i\varphi} e^{-i\omega t}|+\rangle + \cos\theta/2 e^{i\omega t}|-\rangle, \tag{21}$$

where $|+\rangle$ and $|-\rangle$ are the new basis vectors constructed from the linear superposition of the basis vectors $|e\rangle$ and $|g\rangle$, i.e.

$$|+\rangle = \cos\theta/2|g\rangle + \sin\theta/2 e^{-i\varphi}|e\rangle$$
$$|-\rangle = \cos\theta/2|e\rangle - \sin\theta/2 e^{i\varphi}|g\rangle.$$

Here the $\sin\theta/2$ and $\cos\theta/2$ are related to the parameters of the Hamiltonian and is given by

$$\sin\theta/2 = \frac{\omega - \Delta/2}{[2\omega(\omega - \Delta/2)]^{1/2}}, \quad \cos\theta/2 = \frac{\lambda}{[2\omega(\omega - \Delta/2)]^{1/2}} \text{ and } \omega^2 = \lambda^2 + \Delta^2/4.$$

Now the state vector $|\psi(t)\rangle$ undergoes a cyclic evolution over a period $T = \pi/\omega$. So that interval $[0, T]$ the state acquires a phase of π. During the evolution the dynamical phase is $\pi\cos\theta$ and the geometric phase is $\pi(1 - \cos\theta)$. Now we calculate the lengths and the energies of the various curves. The total length of the dynamical curve is just π, whereas the energy associated with the dynamical curve is $\hbar\omega\pi$. The total length of the geometric curve is $[\pi 2\pi(1 - \cos\theta)]^{1/2}$, whereas the associated energy is $\hbar\omega\Omega$, $\Omega = 2\pi\,(1 - \cos\theta)$. Since the geometric length of the curve is equal to the square root of π times the total solid angle Ω subtended by the orbit of motion in a unit sphere, it justifies the name. The total length of the natural curve is equal to $\pi\sin\theta$. The CF for the dynamical curve is $\cos\theta$ and for the geometric curve is $\sin\theta/2$. This shows that the phases are proportional to the length of the curves.

Note that the dynamical phase and the geometric phase calculated on using (12) and (13) are $\pi\cos\theta$ and $\pi(1 - \cos\theta)$ respectively. Using the states $|\psi_\pm(t)\rangle$ we calculate the total length of the curve $t \to |\psi_\pm(t)\rangle$. Now $l(\psi_+)|_0^T$ is given by $\pi^2(1 + \sin\theta)^{1/2}$. Its associated energy is $\hbar\omega\pi(1 + \sin\theta)$. Here, we can also see that the total phase calculated on using (18) is exactly what was expected. Finally $l(\psi_-)|_0^T$ is found to be $[\pi^2(2 + 3\cos\theta - 4\cos\theta)]^{1/2}$. Its associated energy is $\hbar\omega\pi(2 + 3\cos\theta - 4\cos\theta)$. Once again we can calculate the difference phase on using (18). Thus, this simple example demonstrates all the essential features of our formulation of the phases in terms of the length of the curves.

7. Conclusion

To conclude this section, various curves such as the dynamical, geometric and natural curves for an arbitrary cyclic evolution of a quantum system are introduced. Their lengths and energies are defined. It is proved that the phases can be expressed as the integral of the contracted length of some (suitably-defined) curves. Some inequalities among the phases, lengths and energies are shown to be valid. We established the equivalence between the phases and lengths of the curves. Thus wherever we measure the (relative) phase of a state vector undergoing a cyclic evolution, it is nothing but the length of the curve traced by the state vector (with a multiplicative CF). Since the non-adiabatic Berry phase has been measured experimentally, in a sense the contracted length of the geometric curve has been actually measured. So what remain to be measured are the total length of the dynamical curve, the geometric curve and the natural curve.

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