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Universality in the length spectrum of integrable systems

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Abstract. The length spectrum of periodic orbits in integrable hamiltonian systems can be expressed in terms of the set of winding numbers $\{M_1, \ldots, M_f\}$ on the f-tori. Using the Poisson summation formula, one can thus express the density, $\Sigma\delta(T - T_{\mathrm{M}})$, as a sum of a smooth average part and fluctuations about it. Working with homogeneous separable potentials, we explicitly show that the fluctuations are due to quantal energies. Further, their statistical properties are universal and typical of a Poisson process as in the corresponding quantal energy eigenvalues. It is interesting to note however that even though long periodic orbits in chaotic billiards have similar statistical properties, the form of the fluctuations are indeed very different.

Keywords. Length spectrum; periodic orbits; integrable systems; fluctuations; Poisson summation formula.

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The quantal spectra of time-independent Hamiltonian systems exhibit fluctuations that can be classified on the basis of certain broad features of the underlying classical dynamics [1–3]. Typical measures used in these studies are the nearest neighbour level spacing distribution, $P(s)$ and the spectral rigidity, Δ_3. Their behaviour is quite distinct for the two extreme cases of classical flow. In the integrable case, the motion is regular and the quantal spectrum exhibits fluctuations typical of a Poisson process. On the other hand, the fluctuation measures of chaotic systems are adequately modelled by random matrices chosen from ensembles reflecting the presence or absence of anti-unitary symmetries.

Recently, there has been some interest in the statistical properties of the length spectrum of classical periodic orbits as well [4–7]. Clearly, the integrated density of lengths [8] $\Sigma\Theta(T - T_i)$ has a smooth average part superposed with fluctuations which determine the exact staircase. While the mean proliferation of periodic orbits in both integrable and chaotic systems is well established [2], the form of the fluctuations has been investigated only recently [7, 9, 10]. For pseudointegrable billiards [10] as well as for a system comprising of a particle moving on a compact surface of constant negative curvature [9], the analysis is based on a direct inversion of the Gutzwiller trace formula [2]. For chaotic billiards [7], where Maslov indices are orbit dependent, the inversion technique needs to be suitably modified to remove the information of phases altogether. In all three cases however, the average density arises from the zero energy contribution similar to the zero length orbits which give rise to the average density of quantal states. The fluctuations here arise from quantal energies as oscillatory contributions thus establishing the dual relationship between classical periodic orbit lengths and quantal energies.

In the present article, we shall confine ourselves to integrable systems and use a different approach to extract information about the density of lengths. We first show that the length spectrum can be treated on the same footing as the quantal energies for integrable systems since a single expression for the lengths can be obtained in terms of the winding numbers on the tori. We thus obtain the average density as well as the fluctuations using the Poisson summation formula. We also study the statistical properties of these fluctuations and find that the nearest neighbour spacings distribution, $P(s)$ and the spectral rigidity, $\Delta_3(L)$ are universal (for generic integrable systems) and typical of a Poisson process [11].

An f-dimensional integrable system is characterized by the existence of invariant f-tori foliating the entire phase space. Thus there exists a canonical transformation to a set of conjugate variables $(\mathbf{I}, \boldsymbol{\theta})$ such that the hamiltonian is cyclic in the f angles, $\boldsymbol{\theta}$. The time periods (of the periodic orbits) can thus be obtained by solving the set of f equations,

$$\boldsymbol{\omega}(\mathbf{I}) = 2\pi\mathbf{M}/T_{\mathbf{M}}, \tag{1}$$

where $\omega_i = \partial H/\partial I_i$ and $\mathbf{M} = (M_1, M_2, \ldots, M_f)$ denotes the number of windings around the f irreducible circuits. The f-actions, $(I_1, I_2, \ldots, I_f)$ can be eliminated to obtain $T_{\mathbf{M}}$ as a function of the energy, E and $\mathbf{M}$. The procedure is similar to the Böhr–Sommerfeld quantization scheme where $(f-1)$ parameters are eliminated from the f quantization conditions to obtain the energy eigenvalues, $E_{\mathbf{m}}$, in terms of the quantum numbers $\mathbf{m} = \{m_1, m_2, \ldots, m_f)$.

We illustrate this for 2-dimensional separable systems, with potentials of the form, $V(x_1, x_2) = c_1 x_1^{2n} + c_2 x_2^{2n}$. The Hamiltonian in action co-ordinates can be written as $H = \gamma_1 (I_1)^p + \gamma_2 (I_2)^p$ where $p = 2n/(n+1)$ and γ_i are constants which depend on c_i and n. The semiclassical eigenvalues are thus

$$E_{\mathbf{m}} = \hbar^p [\gamma_1 (m_1)^p + \gamma_2 (m_2)^p], \tag{2}$$

while the corresponding time periods obtained using the above prescription are

$$T_{\mathbf{M}} = 2\pi [(M_1/\gamma_1^{1/p})^{p/(p-1)} + (M_2/\gamma_2^{1/p})^{p/(p-1)}]^{(p-1)/p} / p E^{(p-1)/p}. \tag{3}$$

The time periods, $T_{\mathbf{M}}$ (also written as $T(\mathbf{M})$ subsequently) are thus functions of f integer variables like the corresponding quantal eigen-energies. Clearly they can be treated on the same footing. The density of time periods, $\mathrm{d}(T) = \Sigma\delta(T - T_{\mathbf{M}})$, can thus be recast in a set of conjugate integer variables $\{m_1, \ldots, m_f\}$ using the Poisson summation formula [12, 11].

$$\mathrm{d}(T) = \sum_{\mathbf{m}} \int \mathrm{d}^f M \exp(2\pi i \mathbf{M}\cdot\mathbf{m}) \delta(T - T(\mathbf{M})). \tag{4}$$

For large time periods, T (analogous to the semiclassical limit for the energy eigenvalues) the analysis is identical to that of Berry and Tabor (12). The $\mathbf{m} = \mathbf{0}$ term gives rise to the average density, $d_{\mathrm{av}}(T)$ and is equal to

$$d_{\mathrm{av}}(T) = \int \mathrm{d}^f M \, \delta(T - T(\mathbf{M})), \tag{5}$$

an expression that is analogous to the Thomas–Fermi term for the mean density of quantal energies. Similarly, terms with $m_i = 0$ provide the perimeter corrections as discussed by Seligman and Verbaarschot [13]. The rest of the terms in the sum of (4) provides oscillatory corrections and constitute the fluctuating part of the density.

Since part of this work seeks to identify the origin of these fluctuations in non-billiard systems (specifically we wish to investigate the role of quantal energies, if any), we shall deal with systems where both the time periods and quantal energies can be evaluated analytically. To this end, we consider the separable homogeneous potential discussed above for which (2) and (3) give the quantal energies and time periods respectively. Since the system is homogeneous, we shall for simplicity choose $E = (2\pi/p)^{p/(p-1)}$ in the expression for time periods. Using (3) in (5), we thus have

$$d_{\mathrm{av}}(T) = \frac{T}{q(\alpha_1 \alpha_2)^{1/q}} B(1/q, 1/q), \tag{6}$$

where $q = p/(p-1), \alpha_i = (\gamma_i)^{-1/(p-1)}$ and $B(x, y)$ is the beta function. Thus the mean density of time periods is linear in T, irrespective of the value of n in the potential, $V(x_1, x_2)$.

Let us now look at the form of the fluctuations. A transformation to polar co-ordinates, $\alpha_1 I_1^q = r^q \cos^2\phi, \alpha_2 I_2^q = r^q \sin^2\phi$ is helpful in evaluating the integral in (4) with $m_1, m_2 \neq 0$. The Jacobian of this transformation is $2(\alpha_1\alpha_2)^{-1/q} r (\sin\phi\cos\phi)^{(2-q)/q}/q$. For a given $\mathbf{m}$, the integral thus reduces to

$$F_{\mathbf{m}} = \frac{2T}{q(\alpha_1\alpha_2)^{1/q}} \int_0^{\pi/2} \mathrm{d}\phi (\sin\phi\cos\phi)^{(2-q)/q} \exp[i2\pi T\{m_1 \alpha_1^{-1/q} \cos^{2/q}\phi + m_2 \alpha_2^{-1/q} \sin^{2/q}\phi\}], \tag{7}$$

where we have carried out the r integration. For large T, we can now evaluate the ϕ integration by the method of stationary phases. The stationary point occurs at $m_1\alpha_1^{-1/q}\cos^{(2-2q)/q}\phi = m_2\alpha_2^{-1/q}\sin^{(2-2q)/q}\phi$ and the second derivative of the argument in the exponential is

$$i2\pi T \frac{4(1-q)}{q^2} [(m_1\alpha_1^{-1})^{q/(q-1)} + (m_2\alpha_2^{-1/q})^{q/(q-1)}]^{(q-1)/q}.$$

The density of lengths can thus be expressed as

$$\mathrm{d}(T) = d_{\mathrm{av}}(T) + \sum_{\mathbf{m}} A_{\mathbf{m}} \exp(iS_{\mathbf{m}}), \tag{8}$$

where $d_{\mathrm{av}}(T)$ is given by (6),

$$A_{\mathbf{m}} = \frac{\left(\frac{T}{q-1}\right)^{1/2} \{(\gamma_1\gamma_2)^{q-1}(m_1 m_2)^{2-q}\}^{1/2(q-1)}}{[(m_1/\alpha_1^{1/q})^{q/(q-1)} + (m_2/\alpha_2^{1/q})^{q/(q-1)}]^{(3-q)/2q}} \tag{9}$$

and

$$S_{\mathbf{m}} = 2\pi T[(m_1/\alpha_1^{1/q})^{q/(q-1)} + (m_2/\alpha_2^{1/q})^{q/(q-1)}]^{(q-1)/q} - \frac{\pi}{4}. \tag{10}$$

Replacing α_i by $\gamma_i^{-1/(p-1)}$ and q by $p/(p-1)$, it is easy to see that the quantity in the square bracket in (9) and (10) is the quantal eigen-energy of (2) with $\hbar = 1$. Thus fluctuations are indeed due to contributions from (scaled) quantal energies as in pseudointegrable [10] and chaotic [7, 9] systems. Moreover, the average density is due to the zero energy contribution, a fact that is transparent in this derivation for integrable systems.

The statistical properties of these fluctuations form the subject of subsequent discussions. As mentioned earlier, the measures commonly used on the spectrum are the nearest neighbour spacings distribution, $P(s)$ and the special rigidity, $\Delta_3(L)$. The spacings distributions $P(s)$ is defined such that $P(s)\mathrm{d}s$ is the probability of finding adjacent levels (lengths) with spacing between s and $s + \mathrm{d}s$ while the spectral rigidity measures the average mean square deviation of the integrated density of levels (lengths) from the best fitting straight line

$$\Delta(L) = \left\langle \min_{a,b} \frac{d_{\mathrm{av}}}{L} \int_{-L/2d_{\mathrm{av}}}^{L/2d_{\mathrm{av}}} \mathrm{d}\tau [N(T+\tau) - a - b\tau]^2 \right\rangle.$$

For the quantal energies of integrable systems, both $P(s)$ and $\Delta_3(L)$ can be evaluated analytically [1, 3]. Clearly a similar analysis can be carried over for the periodic orbits as well once the relevant scales are defined. The inner scale in the length spectrum is the mean spacing $1/d_{\mathrm{av}}(T)$ while the outer scale ($= L_{\max}/d_{\mathrm{av}}(T)$) is determined by the period of the slowest oscillation (as a function of T) in (8) and is given by $1/(E_0)^{(q-1)/q}$ where E_0 is the ground state energy. We quote here only the final expressions for $P(s)$ [1] and $\Delta_3(L)$ [3] applicable for long periodic orbits

$$P(s) = g(s)\exp\left(-\int_0^s g(u)\mathrm{d}u\right), \tag{11}$$

where

$$g(u) = 1 + \int_{-\infty}^{+\infty} \mathrm{d}k\, e^{iku}\{\phi_D(k) - d_{\mathrm{av}}/2\pi\}$$

and

$$\Delta(L) = \frac{L}{d_{\mathrm{av}}} \int_0^{\infty} \frac{\mathrm{d}y}{y^2} \phi_D(2d_{\mathrm{av}}y/L) G(y), \tag{12}$$

where, $G(y) = 1 - F^2(y) - 3F'^2(y)$, and $F(y) = \sin(y)/y$. Note that (12) holds when $L \ll L_{\max}$ [3]. The function $\phi_D(\varepsilon)$ in both cases is equal to $\langle \Sigma A_{\mathbf{m}}^2 \delta(\varepsilon - \varepsilon_{\mathbf{m}})\rangle$ where $A_{\mathbf{m}}$ and $S_{\mathbf{m}}$ are the amplitude and action defined in (9) and (10) and $\varepsilon_{\mathbf{m}} = 2\pi(E_{\mathbf{m}})^{(q-1)/q}$. The averaging here is over an interval larger than the outer scale and the summation runs over all $\mathbf{m}$. The function $G(y)$ in (12) picks those quantal energies which contribute substantially at a given L. It is almost zero for $y \leqslant \pi/4$ where after it rises monotonically and saturates at a value close to unity for $y > \pi$ [3].

The integrations in (11) and (12) can be carried out once ϕ_D is evaluated. This can be achieved by computing the quantity $\mathscr{F}(\varepsilon) = \int_0^{\varepsilon} \phi_D(\varepsilon')\mathrm{d}\varepsilon' = \Sigma A_{\mathbf{m}}^2$ where the sum now runs over all energies, $E_{\mathbf{m}}$ for which $\varepsilon_{\mathbf{m}} \leqslant \varepsilon$. The summation over m_1 and m_2 in $\mathscr{F}(\varepsilon)$ can be converted to an integral and evaluated quite easily using polar co-ordinates. The limit of r integration is now from 0 to ε while ϕ goes from 0 to $\pi/2$. Using (6) we thus have $\mathscr{F}(\varepsilon) = d_{\mathrm{av}}(T)\varepsilon/2\pi$ and hence $\phi_D(\varepsilon) = d_{\mathrm{av}}(T)/2\pi$ [14] which is identical to the case of quantal energies. Equations (11) and (12) therefore yield $P(s) = e^{-s}$ and $\Delta_3(L) = L/15$ for $L \ll L_{\max}$.

Note that long orbits in chaotic billiards also have identical universal statistical

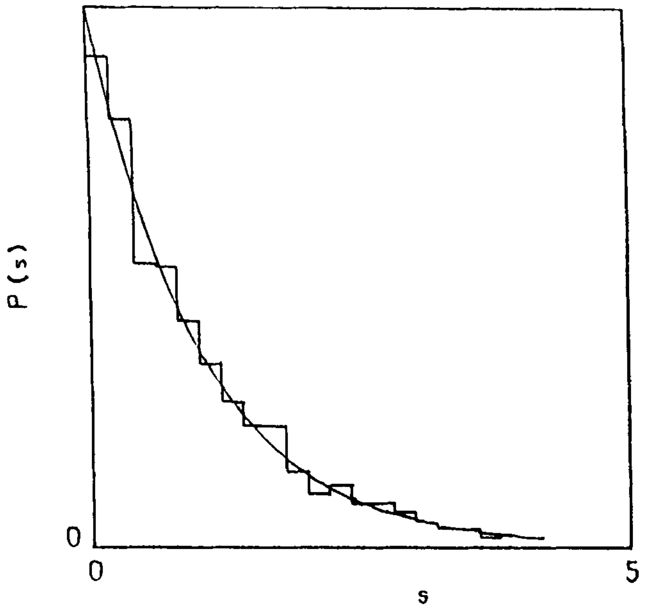

Figure 1. Nearest neighbour spacings distribution, $P(s)$ for the classical time periods. The continuous curve is the Poisson distribution, e^{-s}. For details of the system, see text.

properties [7] even though there are differences in the form of the amplitude of oscillation (see (8) above and ref. [7]).

As pointed out by Biswas *et al* [15] for the energy spectrum, deviations from these universalities do occur when the periodic orbit actions are degenerate. Similar deviations are thus expected here if the actions S_m in (10) are degenerate.

A further consequence of (8) for the density of lengths is the saturation of the spectral rigidity for $L > L_{max}$. This is evident when $\Delta_3(L)$ is expressed as the sum $2\Sigma \frac{A_m^2}{\varepsilon_m^2} G(L\varepsilon_m/2d_{av}(T))$ for integrable systems [3]. For $L > L_{max}$, the function G and hence the spectral rigidity saturates. Note that this is at variance with the observation of Harayama and Shudo (5).

In order to demonstrate our results numerically, we shall consider here a system with $n = 4$, $\gamma_1 = 1{\cdot}21393$ and $\gamma_2 = 0{\cdot}75025$. The time periods have been unfolded in order to compare and characterize the fluctuations. The new sequence thus obtained is used to evaluate the measures, $P(s)$ and Δ_3.

For the nearest neighbour spacings distribution, a sequence of 3000 lengths have been used after eliminating the shortest 3000 orbits. The result is displayed in figure 1. The histogram closely approximates the Poisson distribution, e^{-s}.

Figure 2 shows our result for the spectral rigidity evaluated at the 3500th length and averaged in a stretch containing 1000 lengths. The agreement with the continuous line $(L/15)$ is excellent for $L \leqslant 6$.

Finally, figure 3 shows a plot of the spectral rigidity for the same system evaluated at the 1000th length and averaged in an interval containing 800 lengths. The saturation for $L > 70$ is evident.

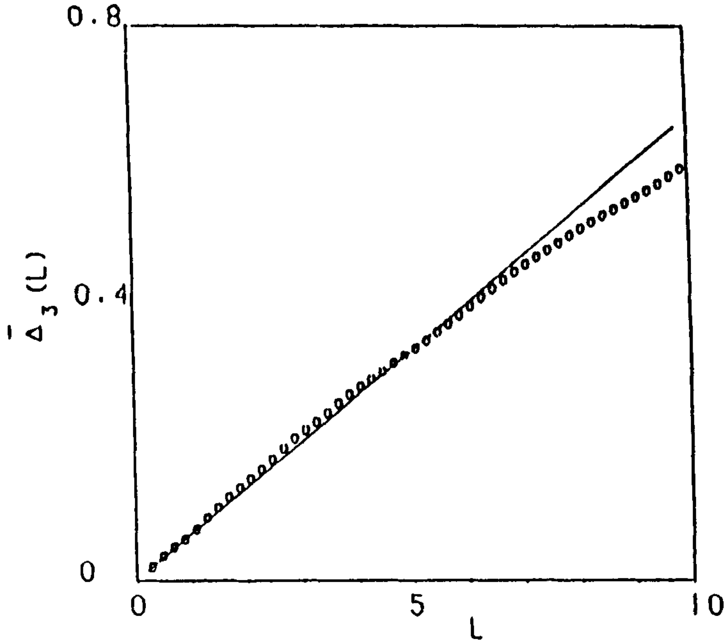

Figure 2. The spectral rigidity, $\bar{\Delta}_3$ for the classical time periods. The continuous line is $L/15$. The agreement is excellent for $L < 6$.

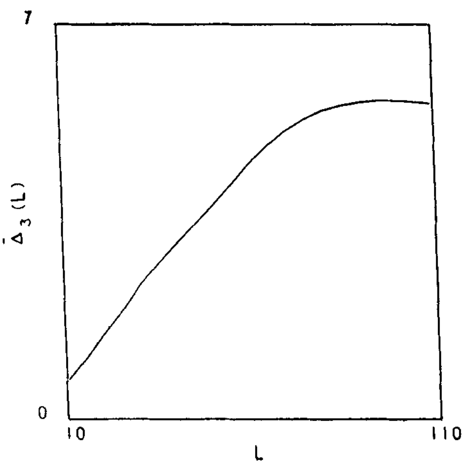

Figure 3. The spectral rigidity for the time periods clearly shows saturation for $L > 70$.

Our studies on other separable potentials confirm the above universalities in the nearest neighbour spacing distribution and the spectral rigidity.

In summary, we have shown that for integrable systems where the length spectrum can be expressed in terms of the winding numbers on the torus, the Poisson summation formula can be used to separate out the average and the fluctuating parts in the density of lengths. Working with separable homogeneous potentials, we explicitly show that the fluctuations are a sum of contributions from quantal energies. Further their statistical properties are typical of a Poisson process as in the case of quantal energy eigenvalues. It is important to note however that while long orbits in chaotic billiards seem to exhibit Poisson fluctuations as well, the form of the amplitude indeed differs from the integrable case.

References

[1] M V Berry and M Tabor, *Proc. R. Soc. London* **A356**, 275 (1977)
[2] M C Gutzwiller, *Chaos in classical and quantum mechanics* (Springer–Verlag, Interdisciplinary Applied Mathematics, 1991)
A M Ozorio de Almeida, *Hamiltonian systems chaos and quantization*, (Cambridge University Press, 1989)
[3] M V Berry, *Proc. R. Soc. London* **A400**, 229 (1985)
[4] M Seiber and F Steiner, *Physica* **D44**, 248 (1990)
[5] T Harayama and A Shudo, *J. Phys.* **A25**, 4595 (1992)
[6] D Biswas and S Sinha, *Phys. Lett.* **A173**, 392 (1993)
[7] D Biswas, *Phys. Rev. Lett.* **71**, 2714 (1993)
[8] The words lengths and time-periods are used in the same sense throughout the text. They are trivially related in homogeneous systems
[9] R Aurich and F Steiner, *Phys. Rev.* **A45**, 583 (1992)
[10] D Biswas and S Sinha, *Phys. Rev. Lett.* **70**, 916 (1993)
[11] D Biswas, BARC/ThPD/DB/Jan'92 (unpublished)
[12] M V Berry and M Tabor, *Proc. R. Soc. London* **A349**, 101 (1976)
M V Berry and M Tabor, *J. Phys.* **A10**, 371 (1977)
[13] T H Seligman and J J M Verbaarschot, *J. Phys.* **A20**, 1433 (1987)
[14] This result has been obtained for billiard systems earlier in [6]
[15] D Biswas, M Azam and S V Lawande, *Phys. Rev.* **A43**, 5674 (1991)