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Ensemble interpretation of the EPR-Bell correlations

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Abstract. The EPR-Bell correlations between the spins of a pair of particles originally in a singlet state are discussed both on the basis of the mathematical formalism of quantum mechanics and the ensemble interpretation. It is shown that the correlations predicted by the mathematical formalism are in agreement with those expected on the basis of the ensemble interpretation, if the electrons are treated as distinguishable particles after they separate and undergo observation. In this case, the correlations are only in partial agreement with a *gedanken* experiment of Mermin on the subject. It is pointed out, however, that agreement with Mermin's conclusions is possible if one treats the electrons as indistinguishable even when they are subjected to observation after separation, though there is no obvious theoretical justification for doing so.

Keywords. Ensemble; locality; singlet; triplet; wavefunction.

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1. Introduction

The paper entitled "Can quantum mechanical description of physical reality be considered complete?" by Einstein *et al* (EPR) [1] has become a classic in the annals of the epistemological foundation of quantum mechanics. The paper brought in sharp focus for the first time the basic problem afflicting quantum mechanics from an interpretational point of view, namely, the fact that the laws of quantum mechanics appear to be either incompatible with local realism [which is a basic tenet of classical physics and which, as defined by Einstein [2], is the premise that "the real states of spatially-separated non-interacting objects are independent"] or incomplete with regard to an individual system.

The heart of the EPR paper is a thought-experiment which, in its Bohm version [3], consists of a pair of spin-half particles in a singlet (spin-zero) state. The pair decays through angular momentum-conserving interactions and the component particles move away in opposite directions. After the interaction between the particles have ceased, the spin of one of the particles in a specified direction $\mathbf{n}$ is measured. It is argued then that the spin of the other particle along $\mathbf{n}$ can be deduced from this measurement (without disturbing that particle) because of the perfect correlation that exists between the spins of two particles in the singlet state. Since the choice of $\mathbf{n}$ is at our disposal and since measurement on particle 1 can in no way affect particle 2 which is spatially separated from particle 1, it follows that all components of the spin of particle 2 have definite values, contrary to the principles of quantum mechanics which permits only one of the components of the spin to have a definite value. One is, therefore, constrained to conclude that either the quantum mechanical wave function, which stipulates perfect correlation between the spins of the two particles, does not provide a complete description of an individual pair or that the assumption of locality is not valid in this case.

The debate that followed the publication of the EPR paper gave rise to the formulation of two schools of interpretation. The Copenhagen or individual system interpretation propounded by Neils Bohr maintains that quantum mechanics is a complete theory in the sense that the wave function provides an exhaustive description of an individual system (in this case, the pair) but demands modification in the Einstein definition of local realism. On the other hand, the ensemble interpretation advocated by Einstein affirms the validity of locality but insists that the wave function refers to an ensemble (of pairs in the singlet state, in the EPR case) and not to an individual system. Vindication of the ensemble interpretation is therefore an indirect validation of locality in quantum mechanics, and vice versa.

2. The EPR-Bell correlations

The EPR thought-experiment is, obviously, beyond the realm of actual experiments. So it cannot be used to settle experimentally the dispute between the advocates and opponents of locality in quantum mechanics. Bell [4] has suggested a modified version of the EPR experiment which at once raises the dispute to the realm of actual experiments. The modifications suggested by Bell consist in:

(i) replacing the single pair of the EPR experiment with an ensemble of pairs.
(ii) replacing the single Stern-Gerlach magnet with one for each particle of the pair.

Thus in the EPR-Bell experiment, we have a pair of Stern-Gerlach magnets D1 and D2 (whose axes can be varied) arranged on the opposite sides of an ensemble of pairs of spin-half particles in the singlet state. D1 measures the spin along a direction specified by the unit vector $\mathbf{a}$ of particles emerging on one side while D2 measures the spin along $\mathbf{b}$ of particles emerging from the other side. The experiment is run until a large number of pairs undergo decay. The recordings of D1 an D2 provide a statistical correlation between the spins of the two particles.

Now according to the quantum theory of angular momentum, the recordings at D1 and D2 can be only either spin-up or spin-down. Therefore, there are four possibilities: spin up–spin up (that is, spin up at D1 and spin up at D2), spin up–spin down, spin down–spin up and spin down–spin down. We denote these respectively by uu, ud, du and dd.

Mermin [5] has described a gedanken experiment which is claimed to demonstrate the results of actual EPR-Bell experiments on the EPR correlations carried out by Alain Aspect and his group [6, 7] at the University of Paris. According to this, the observed correlations are as follows:

(C1) Each of the four cases occur with equal probability and constitutes one-fourth of the total. [In Mermin's demonstration, the proportion of the four cases varies from 22% to 27·7% in an ensemble consisting of 585 individual pairs. For the enormously large number involved in a typical quantum process, the deviation from 25% would be practically nil.]
(C2) However, when $\mathbf{a} = \mathbf{b}$ ($\mathbf{a}$ and $\mathbf{b}$ parallel), a spin up (spin-down) recording at D1 is invariably accompanied by a spin-down (spin-up) recording at D2. That is, there is a perfect correlation between the spins of the two particles.

Mermin contends that these results are in conformity with the mathematical formalism of quantum mechanics and asserts that (C2) constitutes a rebuttal of Einstein's concept of locality. We propose to examine these assertions.

3. Quantum mechanical analysis

Mermin's justification of (C2) is based on the quantum mechanical wavefunction of a pair of spin-half particles in the singlet state, namely

$$|\psi_0\rangle = (2)^{-1/2}[|1-1\rangle - |-11\rangle] \tag{1}$$

where, in $|m_1 m_2\rangle$, m_1 is the value of $\boldsymbol{\sigma}_1$ along the quantization axis while m_2 is that of $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ being the Pauli spin vectors of the two particles constituting the pair. However, this explanation ignores an important feature of the quantum formalism which is repeatedly emphasized by Bohr and which is a key element in his principle of complementarity. This is the fact that observations could seriously affect the state of a quantum system. That is, as remarked by Jordan [8], whereas in classical mechanics measurements merely reveal what exists before the measurement is made, in quantum mechanics "observations not only disturb what has to be measured, they produce it." Consequently, it is necessary to make a distinction between the properites of a system contained in its wavefunction (when the system is closed) and the properties that will be revealed in an experiment (when the system is open). We can illustrate this point using the simple example of a spin-half particle:

Consider the particle in a spin-up state quantized along the z-axis. We will denote the state by the ket $|z{:}1\rangle$. This state has the property

$$\sigma_z|z{:}1\rangle = |z{:}1\rangle, \tag{2}$$

$$\langle z{:}1|\sigma_x|z{:}1\rangle = 0. \tag{3}$$

These equations signify that the state $|z{:}1\rangle$ is characterized by a definite value for σ_z namely $+1$, which can be measured, and an average value of zero for σ_x which, however, cannot be measured (i.e., σ_x has no definite value in $|z{:}1\rangle$). In fact, any experimental arrangement designed to measure σ_x will change the state of the particle from $|z{:}1\rangle$ to $|x{:}\pm 1\rangle$. This is the meaning of the remark quoted above as due to Jordan. Observation of σ_x not only disturbs the state $|z{:}1\rangle$ in which σ_x has no definite value, but produces the state $|x{:}1\rangle$ or $|x{:}-1\rangle$ in which it has a definite value. Here we see another peculiarity of the quantum formalism: it cannot say for certain which of the two states, $|x{:}1\rangle$ or $|x{:}-1\rangle$, will be produced in any particular observation. In order to see what exactly would be the result of the experiment, we should adopt the following procedure:

Choose a basis to construct the matrix of σ_x. Then diagonalize this matrix to obtain the eigenvalues and eigenvectors of σ_x. Since $|z{:}1\rangle$ alone does not form a basis, we have to include the vector $|z{:}-1\rangle$ also. Then, the matrix of σ_x is $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and the eigenvalues are ± 1 with corresponding eigenvectors given by

$$|x{:}1\rangle = (2)^{-1/2}[|z{:}1\rangle + |z{:}-1\rangle], \tag{4a}$$

$$|x{:}-1\rangle = (2)^{-1/2}[|z{:}1\rangle - |z{:}-1\rangle]. \tag{4b}$$

We could invert (4a, 4b) to obtain

$$|z{:}1\rangle = (2)^{-1/2}[|x{:}1\rangle + |x{:}-1\rangle], \tag{5a}$$

$$|z{:}-1\rangle = (2)^{-1/2}[|x{:}1\rangle - |x{:}-1\rangle]. \tag{5b}$$

These equations imply that, if we prepare a large number of spin-half particles in the state $|z{:}1\rangle$ and then make a measurement of σ_x, we will get the value $+1$ for about half the cases and -1 for the other half. But we note that the same will be the case if the particles are prepared in the $|\mathbf{z}{:}-1\rangle$ state instead of the $|z{:}1\rangle$ state. Therefore, the measured values of σ_x do not reflect any specific (or, exclusive) property of either $|\mathbf{z}{:}1\rangle$ or $|z{:}-1\rangle$, but represents merely the property of an ensemble of spin-half particles. This conclusion is further confirmed by the fact that there would be no change in the observed values of σ_x if we replace the initial states (quantized along the z-axis) with those quantized along the y-axis; and that the same proportion of spin-up and spin-down states are obtained if we observe σ_y instead of σ_x. The conclusions of the foregoing considerations may be summarized as follows:

(a) Equation (2) represents a specific property of the state $|z{:}1\rangle$. This property can be verified by measuring σ_z. That is, measurement of σ_z will not change the state of the system.

(b) Equation (3) may be interpreted in two ways:

(i) It represents a (non-exclusive) property of $|z{:}1\rangle$. The equation is exact in this sense (that is, the average value of σ_x is exactly zero in the state $|z{:}1\rangle$) but cannot be verified by observing σ_x.

(ii) It represents the property of an ensemble of spin-half particles and tells that in a measurement of σ_x (or any other component of $\boldsymbol{\sigma}$) $+1$ and -1 values will each be realised half of the time, provided the number of particles involved is large enough. The latter condition emphasizes that the property is a statistical one.

The first one of the above interpretations refer to the property of the wavefunction $|z{:}1\rangle$ (closed system), and the second one to the property of a spin-half (open) system that will be revealed in an experiment. Bohr has been emphasizing the first interpretation and Einstein insisting on the second. But it is obvious that we need both (in their distinct domains) if quantum mechanics is to be regarded not only as a self-consistent physical theory, but also as one that can account for actual observations.

In short, there are two types of properties implicit in the wavefunction of an individual quantum system. Those which can be directly observed (without affecting the wavefunction) and those which cannot be so observed but are nevertheless required by the quantum mechanical formalism. An observational realisation of the latter is, however, possible in relation to an ensemble.

Let us look at (1) in the light of the above considerations. To be more specific, let us rewrite the equation in the form,

$$|\psi_0\rangle = (2)^{-1/2}[|z{:}1-1\rangle - |z{:}-11\rangle]. \tag{1a}$$

The quantum mechanical properties of this state are contained in the equations,

$$\{(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\mathbf{n}\}|\psi_0\rangle = 0 \tag{6}$$

$$(\boldsymbol{\sigma}_1\cdot\mathbf{n})(\boldsymbol{\sigma}_2\cdot\mathbf{n})|\psi_0\rangle = -\mathbf{n}^2|\psi_0\rangle = -|\psi_0\rangle \tag{7}$$

where **n** is a unit vector. These follow from the relationships,

$$(\boldsymbol{\sigma}_1\cdot\mathbf{n})|\psi_0\rangle = n_z|\psi_{10}\rangle + (2)^{-1/2}\{n_+|\psi_{1-1}\rangle - n_-|\psi_{11}\rangle\} \tag{8a}$$

$$(\boldsymbol{\sigma}_2\cdot\mathbf{n})|\psi_0\rangle = -n_z|\psi_{10}\rangle + (2)^{-1/2}\{n_-|\psi_{11}\rangle - n_+|\psi_{1-1}\rangle\} \tag{8b}$$

with

$$|\psi_{10}\rangle = (2)^{-1/2}[|z{:}1-1\rangle + |z{:}-11\rangle] \tag{9a}$$

$$|\psi_{1-1}\rangle = |z{:}-1-1\rangle \tag{9b}$$

$$|\psi_{11}\rangle = |z{:}11\rangle \tag{9c}$$

and

$$n_\pm = (n_x \pm i n_y). \tag{10}$$

The singlet state of the pair is thus characterised by (i) a total spin equal to zero [eq. (6)] and (ii) a perfect correlation between the spins of the two particles comprising the pair [eq. (7)]. The second characteristic means that a spin-up (spin-down) state of one particle is accompanied by a spin-down (spin-up) state for the other particle. However, (8a, b) show that these characteristics of the singlet state cannot be verified by measuring the spins of the individual (component) particles just as characteristic (3) of the state $|z{:}1\rangle$ cannot be verified by measuring σ_x. The reason in both cases is the same, the concerned observables have no definite values in the state under consideration (that is, the states are not eigenvectors of these observables).

4. Experiment to measure individual spins

In spite of the conclusions arrived at above, if one sets up an experiment to observe the spins of the individual particles, Jordan's dictum (quoted in the previous section) will rule the results. In other words, the results of the experiment would reflect only the general properties of an ensemble of pairs of spin-half particles, and not an exclusive property of the singlet system. In order to deduce the results of the experiment, we can adopt the same procedure as the one followed in the measurement of σ_x. That is, construct the matrix of $[(\boldsymbol{\sigma}_1\cdot\mathbf{a})(\boldsymbol{\sigma}_2\cdot\mathbf{b})]$ in a basis corresponding to the pair, and diagonalise the matrix to get the eigenvalues and eigenvectors.

We choose as basis states the vectors (in the notation of §3) $|z{:}1-1\rangle$, $|z{:}11\rangle$, $|z{:}-11\rangle$ and $|z{:}-1-1\rangle$, which are the eigenvectors of the observables corresponding to the components of the spins of the individual particles. The matrix can be calculated by writing

$$(\boldsymbol{\sigma}_1\cdot\mathbf{a}) = \sigma_{1z}a_z + (1/2)[\sigma_{1+}a_- + \sigma_{1-}a_+] \tag{11}$$

and similarly $(\boldsymbol{\sigma}_2\cdot\mathbf{b})$, and using the relationships,

$$\langle z{:}m_3m_4|\sigma_{1z}\sigma_{2z}|z{:}m_1m_2\rangle = m_1m_2\delta_{m_1m_3}\delta_{m_2m_4}, \tag{12a}$$

$$\langle z{:}m_3m_4|\sigma_{1z}\sigma_{2\pm}|z{:}m_1m_2\rangle = m_1\{(1\mp m_2)(3\pm m_2)\}^{1/2}\delta_{m_1m_3}\delta_{m_2\pm 2,m_4}, \tag{12b}$$

$$\langle z{:}m_3m_4|\sigma_{1\pm}\sigma_{2z}|z{:}m_1m_2\rangle = m_2\{(1\mp m_1)(3\pm m_1)\}^{1/2}\delta_{m_1\pm 2,m_3}\delta_{m_2m_4}, \tag{12c}$$

$$\langle z{:}m_3m_4|\sigma_{1\pm}\sigma_{2\pm}|z{:}m_1m_2\rangle = \{(1\mp m_1)(1\mp m_2)(3\pm m_1)(3\pm m_2)\}^{1/2}\delta_{m_1\pm 2,m_3}\delta_{m_2\pm 2,m_4} \tag{12d}$$

and

$$\langle z{:}m_3 m_4 | \sigma_{1\pm} \sigma_{2\mp} | z{:}m_1 m_2 \rangle = \{(1 \mp m_1)(1 \pm m_2)(3 \pm m_1)(3 \mp m_2)\}^{1/2} \delta_{m_1 \pm 2, m_3} \delta_{m_2 \mp 2, m_4} \tag{12e}$$

We consider three different cases: **a**// **b**//**z**, **a**//**b** ⊥ **z** and **a** ⊥ **b**//**z**. In all these cases the eigenvalues are + 1 and − 1, each of degeneracy 2 (This could, of course, be deduced without diagonalising the matrix, but the matrix diagonalization confirms these values). The wavefunctions are listed in Appendix A.

These wavefunctions have been obtained subject to the constraint that they be also eigenvectors of $(\sigma_1 \cdot \mathbf{a})$ and $(\sigma_2 \cdot \mathbf{b})$. We feel that, the conditions of the experiment require this. So the wavefunctions are labelled by the eigenvalues of both $(\boldsymbol{\sigma}_1 \cdot \mathbf{a})(\boldsymbol{\sigma}_2 \cdot \mathbf{b})$ and $(\boldsymbol{\sigma}_1 \cdot \mathbf{a})$. Thus $|+ -\rangle$ is an eigenvector belonging to eigenvalue + 1 of $(\boldsymbol{\sigma}_1 \cdot \mathbf{a})(\boldsymbol{\sigma}_2 \cdot \mathbf{b})$ and eigenvalue − 1 of $(\boldsymbol{\sigma}_1 \cdot \mathbf{a})$.

Since the initial (singlet) state itself does not occur as one of the eigenvectors, it may be concluded that all the four eigenvectors, $|+ +\rangle$, $|+ -\rangle$, $|- +\rangle$ and $|- -\rangle$, would be realized in the experiment with equal probability. Thus, Mermin's conclusion (Cl) is justified. For the same reason, conclusion (C2) would not be consistent with the results.

Let us elaborate this point:

Mermin's conclusion (C2) is that whenever **a** is parallel to **b**, a spin-up recording at D1 must be accompanied by a spin-down recording at D2 and vice versa. In our formalism, this requires that either the singlet state ψ_0 [eq. (1a)] should continue to be an eigenvector or else only the eigenvectors $|- +\rangle$ and $|- -\rangle$ listed in Appendix A should be realized [remember that, in the states $|- +\rangle$ and $|- -\rangle$, $(\boldsymbol{\sigma}_1 \cdot \mathbf{a})$ and $(\boldsymbol{\sigma}_2 \cdot \mathbf{b})$ have eigenvalues differing in sign, whereas in the states $|+ +\rangle$ and $|+ -\rangle$ these operators have the same eigenvalue]. Neither of these conditions are satisfied, for, we have no reason to rule out the other two eigenvectors $|+ +\rangle$ and $|+ -\rangle$.

5. The ensemble interpretation

It is easy to see that the result of the above quantum mechanical analysis of the outcome of measurements on the spins of the individual particles are in agreement with the ensemble interpretation of the wavefunction $|\psi_0\rangle$ [eq. (1a)] of the singlet pair. For, according to this interpretation, (1a) does not represent an individual pair, but an ensemble of pairs of spin-half particles in the singlet state such that the number of pairs with particle 1 in the spin-up state and particle 2 in the spin-down state is the same as the number with particle 1 in the spin-down state and particle 2 in the spin-up state. Thus, in the experiment, the number with spin-up recording at D1 accompanied by spin-down recording at D2 will be the same as the number with spin-down recording at D1 accompanied by spin-up recording at D2. But it is essential for the validity of this ensemble interpretation that every recording of spin-up (spin-down) at D1 should not be associated with spin-down (spin-up) recording at D2. For, if it were so, then the spin-correlation represented by (1a) would be applicable to every single observation, rendering thereby the ensemble interpretation redundant. Hence there should be spin-up (spin-down) recording at D1 accompanied by spin-up (spin-down) recording at D2.

However, as we explained in § 3, the predicted results of observation on the individual spins do not invalidate the individual system interpretation of the wavefunction. The two interpretations merely refer to two different aspects of the quantum system. Thus, it appears that Mermin's conclusion (C2), if true, will not only invalidate locality but also certain basic mathematical aspects of the quantum formalism as they relate to the wavefunction of a quantum system.

6. An explanation of (C2)

It should be emphasized that in the foregoing analysis we have treated the two electrons of the pair after separation and on being subjected to observation as distinguishable particles. The logic of the experimental set up demands this: the electrons are far apart; we can tell if the electron registering at a particular detector is spin up or down. However, if (C2) is really an experimental fact (as claimed by Mermin—though we would like it to be confirmed by two or three indeoendent experimental groups), then the theory has to account for it.

It is possible to get agreement with (C2) if we treat the electrons as indistinguishable. This means that we should use properly symmetrised wavefunctions of the pair [that is, singlet and triplet combinations—(1a) and (9a–c)] as basis states in place of the unsymmetrized ones. The matrix of $(\boldsymbol{\sigma}_1\cdot\mathbf{a})(\boldsymbol{\sigma}_2\cdot\mathbf{b})$ in this case is given in Appendix B. It is obvious that a pair initially in the singlet state will continue to be in that state if $\mathbf{a}$ is parallel to $\mathbf{b}$. So (C2) is accounted for. But here the problem is how to theoretically justify treating the two electrons as indistinguishable in the given situation.

7. Conclusion

We have discussed, in this paper, the EPR-Bell correlations from the point of view of the ensemble interpretation as well as on the basis of the mathematical formalism of quantum mechanics. We have shown that the correlations predicted by the mathematical formalism are in agreement with those expected on the basis of the ensemble interpretation if we treat the electrons of the pair as distinguishable particles after the decay. In this case the correlations are only in partial agreement with the conclusions of Mermin represented in a thought-experiment. In particular, the conclusion that the spins of the two electrons would be correlated as in the singlet state whenever the axes of the two detectors are set parallel, is not borne out. Agreement with Mermin's conclusions is, however, possible if we treat the electrons as indistinguishable even after they are subjected to observation on separation; though there appears to be no theoretical justification for doing so.

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Appendix A

Listed here are the common eigenvectors of $(\boldsymbol{\sigma}_1\cdot\mathbf{a})\,(\boldsymbol{\sigma}_2\cdot\mathbf{b})$, $(\boldsymbol{\sigma}_1\cdot\mathbf{a})$ and $(\boldsymbol{\sigma}_2\cdot\mathbf{b})$ for three cases. The eigenvectors are identified by the eigenvalues of $(\boldsymbol{\sigma}_1\cdot\mathbf{a})\,(\boldsymbol{\sigma}_2\cdot\mathbf{b})$ and $(\boldsymbol{\sigma}_1\cdot\mathbf{a})$.

Thus $|-+\rangle$ denotes an eigenvector belonging to eigenvalue -1 of $(\boldsymbol{\sigma}_1\cdot\mathbf{a})(\boldsymbol{\sigma}_2\cdot\mathbf{b})$ and eigenvalue $+1$ of $(\boldsymbol{\sigma}_1\cdot\mathbf{a})$.

case 1. $\mathbf{a}//\mathbf{b}//\mathbf{z}$ $(a_z = b_z;\ a_- = a_+ = 0)$

$$\begin{aligned}|++\rangle &= |11\rangle,\\ |+-\rangle &= |-1-1\rangle,\\ |-+\rangle &= |1-1\rangle,\\ |--\rangle &= |-11\rangle.\end{aligned} \tag{A1}$$

case 2. $\mathbf{a}//\mathbf{b}\perp\mathbf{z}$ $(a_z = b_z = 0;\ a_-a_+ = 1)$

$$\begin{aligned}|++\rangle &= (1/2)[a_-|11\rangle + |1-1\rangle + |-11\rangle + a_+|-1-1\rangle],\\ |+-\rangle &= (1/2)[-a_-|11\rangle + |1-1\rangle + |-11\rangle - a_+|-1-1\rangle],\\ |-+\rangle &= (1/2)[-a_-|11\rangle + |1-1\rangle - |-11\rangle + a_+|-1-1\rangle],\\ |--\rangle &= (1/2)[a_-|11\rangle + |1-1\rangle - |-11\rangle - a_+|-1-1\rangle].\end{aligned} \tag{A2}$$

case 3. $\mathbf{a}\perp\mathbf{b}//\mathbf{z}$ $(a_z = 0;\ b_z = 1;\ a_-a_+ = 1;\ b_-b_+ = 0)$

$$\begin{aligned}|++\rangle &= (1/2)^{1/2}[|11\rangle + a_+|-11\rangle],\\ |+-\rangle &= (1/2)^{1/2}[|1-1\rangle - a_+|-1-1\rangle],\\ |-+\rangle &= (1/2)^{1/2}[|11\rangle - a_+|-11\rangle],\\ |--\rangle &= (1/2)^{1/2}[|1-1\rangle + a_+|-1-1\rangle].\end{aligned} \tag{A3}$$

Appendix B

Here we give the matrix of $(\boldsymbol{\sigma}_1\cdot\mathbf{a})(\boldsymbol{\sigma}_2\cdot\mathbf{b})$ for $\mathbf{a}//\mathbf{b}$ when the basis states are ψ_{00}, ψ_{11}, ψ_{10} and ψ_{1-1}. The rows and columns of the matrix are labelled in this order.

$$\begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & (1-a_+a_-) & \sqrt{2}a_-a_z & a_-^{\,2}\\ 0 & \sqrt{2}a_+a_z & (2a_+a_- - 1) & -\sqrt{2}a_za_-\\ 0 & a_+^{\,2} & -\sqrt{2}a_za_+ & (1-a_+a_-)\end{pmatrix}$$

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