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WKB approximation in complex time II

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Abstract. The non-perturbative method, developed recently, of WKB approximation in complex time is applied to some known curved space time. Three cases namely (1) static in and out region, (2) non-static in and out region, (3) static in and non static out region are considered here. We find non-trivial particle production corresponding to the quantum vacuum definition of Castagnino and Mazzitelli.

Keywords. Multiple reflection; particle production; curved space time.

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1. Introduction

The particle production by a time dependent field is an active area of research both in quantum electrodynamics and quantum cosmology. The primary work in this direction dates back to Schwinger [1] who calculated Im L_{eff} (needed in defining probability of particle production) for particles moving in electric and magnetic fields, and the field of a plane electromagnetic wave. Subsequently many authors [2–8] have considered the particle production in various frameworks. In quantum cosmology the particle production plays an important role in the inflationary model of the universe. In this model though the universe classically turns back from a finite radius to an expanding mode, quantum corrections that occur at Planck length do not lead to bounce off at finite radius and re-expand to infinity. The universe may not reexpand at all or if it expands it does not settle down to Friedman solution, rather it explodes with a de Sitter mode with

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2], \tag{1}$$

where $a(t) = \exp(At)$; $A > 0$. To have an exit from the inflationary scenario, one requires damping so that $a(t)$ settles down to steady Friedman mode. It is suggested that in isotropic cosmologies damping comes from particle production and it is due to time dependent change in curvature or matter fields in the model. This in turn is related to the cosmological constant problem. In all standard models of inflationary universe, it is impossible to demand zero vacuum energy in both the symmetric and asymmetric phase. Practically vacuum energy is large enough i.e., we have a large cosmological constant whereas the present limit of cosmological constant fixed by Hubble's measurement is $|\Lambda|/m_p^2 < 10^{-120}$, where m_p is the Planck mass. It is hoped that in expanding universe particle production might cause a large $\Lambda_{initial}$ to arrive at the value $\Lambda_{obs}/m_p^2 < 10^{-120}$ satisfactorily. This idea is as follows. In curved space time when one deals with motion of particles in a gravitational background, one requires to define 'in' and 'out' vacuua corresponding to given gravitational background. In such a situation particle production might occur as the particle evolves from 'in' vacuum to 'out' vacuum due to a change in vacuum energy. This is turn, can be regarded as a change in the effective cosmological constant as the particle production proceeds. In view of the importance of particle production as mentioned above, it is necessary to look at its scenario in expanding space time. The present paper is an attempt in this direction.

In this paper we will concentrate mainly on the particle production in some models of expanding universe i.e., $\dot{a}/a > 0$ within the framework of recently proposed WKB approximation in complex time [9: referred to as I] to settle the role of vacuua, a delicate object in curved space time. The standard procedure in dealing with particle production is to evaluate the Bogolubov mixing transformation and classify the vacuum. The energy momentum of the created particle is evaluated by computing the energy differences corresponding to the two sets of vacuua, namely adiabatic and conformal vacuua. In curved space time the appropriate definition of vacuum is still not well resolved. Moreover one is not able to calculate $\langle T_{\mu\nu} \rangle$ more than one loop approximation satisfactorily. In Bogolubov transformation technique, out of the many possible solutions only two particular sets are chosen to fix the vacuua and hence the Bobolubov coefficients α_k and β_k are fixed. A non-zero β_k implies particle production with a mixing of positive and negative energy states. The procedure does not elucidate clearly the role of paths especially the complex paths of scattering states as well as the origin of thermal spectrum. More specifically the role of tunnelling paths i.e., paths in imaginary time is not exemplified by Bogolubov transformation technique. The complex paths are necessary to explain the particle production in curved space time.

To bypass this difficulty, we take recourse to a non perturbative method of WKB approximation in complex time, recently developed by us [9, 10]. In most problems e.g., problems dealing with motion of particles in strong time varying electromagnetic field, the Wheeler–DeWitt equation in quantum cosmology [11], transport equation in quark gluon plasma based on Wigner representation [12], the temporal equation takes the form of one dimensional Schrödinger equation, not in space but in time, viz.,

$$\frac{d^2\psi}{dt^2} + [w^2 - V(t)]\psi = 0, \tag{2}$$

where w plays the role of energy and $V(t)$ is a time dependent potential.

Knoll and Schaeffer [13, 14] and Schrempp and Schrempp [15] developed the method of complex reflection using semiclassical paths in space. We extend the method to the time variable in the form of the complex multiple reflection in time [9, 10] treating $V(t)$ as a function of complex time t.

The WKB approximation for complex trajectories accounts for contribution of the order of $\exp(-c/\hbar)$ to the usual WKB wave which would vanish in the classical limit. The WKB approximation generalised to complex time turns out to reproduce quantitatively all quantum mechanical effects and remain accurate in cases when the potential $V(t)$ varies rapidly over a distance of wavelength or less.

In some problems particle production takes place by non-local barrier penetration. In WKB approximation this non-local behaviour is well described by non-local functional $\int [w^2 - V(t)]^{1/2} dt$. Moreover, the complex path approach takes into account the behaviour of tunnelling path (mainly in Euclidean space) that is barely needed for mixing of positive and negative energy states. One loop correction, basically an order $\hbar$ approximation, breaks down when the quantum correction is large enough compared to classical contribution. Therefore it is necessary to look at the particle production through semiclassical WKB approximation, and if it is unimportant in WKB approximation it is unlikely to be important in an improved approximation [2]. In wavefunction or classical path approaches it is found by many authors that the region near the turning points (bounces in the problem considered) is important, whereas the WKB approximation (real paths) is needed to show that the region away from the points is negligible.

In § 2 we briefly outline the method of complex multiple reflection in time. The role of vacuum has been discussed in § 3. In § 4 we apply our method to calculate pair production amplitude in some known space times. We discuss three cases viz., the static in and out region; non static in and out region and finally static in and non-static out region. The consequences of the results are then discussed. Concluding remarks are given in § 5.

2. WKB approximation in complex time

Let us start with the one dimensional Schrödinger equation in time i.e., equation (2) and generalize the result of Schrempp and Schrempp [14] in complex t-plane where $V(t)$ is treated as a function of complex t. In this approach one starts at a real point and ends at another real point, but is complex in between. In our work we consider two cases with

$$\Omega(t) = [w^2 - V(t)]^{1/2}, \tag{3}$$

having one and two turning points, real or complex. The standard WKB approximation with real t is written as [15]

$$\psi = \frac{C_1}{[\Omega(t)]^{1/2}} \exp \int i\Omega(t)\mathrm{d}t + \frac{C_2}{[\Omega(t)]^{1/2}} \exp \int (-i)\Omega(t)\mathrm{d}t. \tag{4}$$

To proceed with WKB approximation in complex time we define

$$S(t,t') = \int_{t'}^{t} \Omega(t)\mathrm{d}t = S(t) - S(t'). \tag{5}$$

In (5), $S(t)$ refers to the value of the integral evaluated at the limit t i.e.,

$$S(t) = \left[\int \Omega(t)\mathrm{d}t \right]_t. \tag{6}$$

Treating $\Omega(t)$ as a function of complex t the turning points are determined from

$$\Omega(t) = [w^2 - V(t)]^{1/2} = 0. \tag{7}$$

The boundary conditions are chosen such that

$$\begin{aligned} &\psi \sim \exp[iS(t\,t_1)] &&; \quad t \to -\infty \\ &\psi \sim \exp[iS(t\,t_1)] + iR\exp[-iS(t\,t_1)]; && \quad t \to \infty \end{aligned} \tag{8}$$

where R is called reflection amplitude. In (8) t is real but the turning points t_1 and t_2 are complex. When we consider the pair production as a consequence of reflection in time, the reflection coefficient is identified as pair production amplitude. The basis of our method rests on Feynmann prescription [9] of negative energy particle solution propagating backward in time as being equivalent to positive energy antiparticle solution moving forward in time. The idea of considering particle production as a process of reflection in time was also pursued by Cornwall and Ticktopoulos [17], albeit in a different context. They view pair production as Klein paradox like situation, not in space but in time. Our approach is a generalization of the Klein paradox like situation, not in space but in time, proposed by Cornwall and Ticktopoulos [17]. In I we have established connection between reflection in time and pair production namely

$$|\text{pair production amplitude}|^2 = |\text{reflection amplitude}|^2. \tag{9}$$

In other words, the particle production amplitude, using complex semiclassical method yields wavefunction identical to that obtained by generalized WKB approximation.

Case A: One turning point

For one complex turning point t_1, the complex semiclassical approximation leads to the wavefunction [9]

$$\psi(t) \underset{t\to\infty}{\sim} \exp[iS(t\,t_0)] + (-i)\exp[iS(t_1\,t_0)]\exp[-iS(t\,t_1)] \tag{10}$$

where we have dropped the WKB pre-exponential factor for convenience. In (10) the complex turning point is determined from (7) and $S(t\,t')$ is defined as in (5). Equation (10) is interpreted as follows. A wave starting from $t = t_0$ and ending at t can be considered to be composed of (i) a wave starting at $t = t_0$, moving left reaches t and (ii) a wave starting at $t = t_0$ reaches the complex reflection point t_1 and turns back towards t. The factor $(-i)$ in (10) has been introduced to ensure reflection at t_1.

Equation (10) leads to the pair production amplitude [9]

$$R = -\exp[2iS(t_1)]. \tag{11}$$

Case B: Two turning points

For two turning points t_1 and t_2 in the complex t plane the resulting multiple reflection series is given by [9, 10]

$$\psi(t) \underset{t\to\infty}{\sim} \exp[iS(t\,t_0)] + (-i)\exp[iS(t\,t_0) - iS(t\,t_1)] \sum_{\mu=0}^{\infty} [-i\exp\{iS(t_1\,t_2)\}]^{2\mu} \tag{12}$$

The interpretation of (12) is as follows. The classical trajectories building up the quantum mechanical waves are (i) the direct (real) trajectory from t_0 to t represented by the first term in (12) and (ii) the trajectory from t_0 returning to t after complex reflections between t_1 and t_2 leading to geometrical series $\Sigma_{\mu=0}^{\infty}$ in (12). In the usual WKB approximation with real semiclassical path we have only the first term $\exp[iS(t\, t_0)]$ in (12). But in the case under discussion a second term makes its presence due to contribution from reflection. Equation (12) leads to the reflection coefficient [9]

$$R = -\frac{\exp[2iS(t_1)]}{1 + \exp[2iS(t_1 t_2)]}. \tag{13}$$

In the expressions for pair production amplitude R in (11) and (13) we have overlooked contribution from t_0 limit. This is justified in view of the following. In our early works [9, 10] we have established the equivalence between this method and the Bogolubov transformation technique [18, 19] that is generally used in cosmological particle creation problems. One may recall that while calculating the pair production amplitude through Bogolubov technique we form two sets of normalized modes $\{\bar{u}_p, \bar{u}_p^*\}$ and $\{u_p, u_p^*\}$ related by

$$\bar{u}_p = \alpha_p u_p + \beta_p u_p^*. \tag{14}$$

Our R is related to the Bogolubov coefficients α_p and β_p as $|R| \sim \left|\frac{\beta_p}{\alpha_p}\right|$. Since the same contributions from the t_0 limit occurs in both α_p and β_p they cancel out and hence do not occur in R.

The method we are advocating has many attractive features. In this new method the exact solution of the field equation is not required explicitly and also propagators that make the situation very complicated. Our method requires calculation of simple integrals (involving branch points) like (5) and (6) at the turning points. In spite of the extreme simplicity of our method we get results [9, 10, 20, 21] which coincide with the standard results suggesting that the complex multiple reflection technique in time is a valid description for calculating the pair production amplitude. A very important aspect of our method is that while evaluating the reflection amplitude the behaviour of the solution near the turning point is not required, and we take the contributions of classical paths and tunnelling paths (i.e., imaginary paths in time) correctly which will be justified through calculations that follow. The appropriateness of our method will be clarified from the work of Neville [2]. However they were unable to tackle the contributions of tunnelling paths correctly, though they mentioned the usefulness of complex paths.

Another significant aspect of our method is that the pair production amplitude depends on the bounce points (i.e., turning points) which lie in the complex plane. Basically we get contributions from disconnected region i.e., from the euclidean space.

We have employed our technique to the case of pair production in time dependent electromagnetic field [10, 20] and obtained results which coincide with that of Barut and Duru [22]. We have shown elsewhere [10, 20] that our technique leads exactly to the result of Cornwall and Ticktopoulos [17]. We have further applied the method of complex multiple reflection in time to the pair production scenario in two dimensional Milne universe and reproduced standard results [9, 21]. In quantum cosmology we have calculated the wavefunction of the universe identifying the pair production wavefunction as wormhole solutions in Euclidean space. We in this paper, test how our method works when we consider different space times to find pair production amplitude.

3. Role of vacuua

Before we pass on to the calculation of pair production amplitude in curved space time it is necessary to specify the choice of vacuum in the calculation. In curved space time the correct choice of vacuum is still an open question. It has been shown that a reasonable quantum vacuum can be defined in Bianchi type I universes, at the cosmological singularity and at the asymptotic far future region provided the initial expansion is not violent. The reader is referred to the works of Castagnino and others [23–26] for the vacuum definition in curved space time. These authors prescribe a vacuum choice as

$$\psi_k^{\text{in}} = \begin{cases} \dfrac{1}{(2k)^{1/2}} \exp(-ik\eta) ; & \eta \leqslant 0 \\ \alpha_k \phi_k^{\text{out}} + \beta_k \phi_k^{*\,\text{out}} ; & \eta > 0 \end{cases} \tag{15}$$

where the Bogolubov coefficient $|\beta_k|^2$ measures the pair production between $t = 0$ and $t = \infty$; ϕ_k^{out} is given by the WKB approximation solution of the field equation and η is a conformal time defined in (17).

Our definition is basically equivalent to the above definition of vacuum. However we extend the idea of Cornwall and Ticktopoulos [9, 17] as follows. At $t = -\infty$ there is no particle but at $t = +\infty (t > t_a)$ due to reflection in time there is a particle moving forward in time and an antiparticle moving backward in time; where t_a is the reflection point. As the particle-antiparticle definition in Minkowski time is not valid in curved space time we adopt Parker's definition [27] so that particle-antiparticle solution are now replaced by $\exp(\pm iS)$ where S is obtained through WKB solution. With this definition our reflection coefficient will be equivalent to $|\beta_k/\alpha_k|$ of Castagnino and others [23–26]. The equivalence of the two approaches is quite convincing for the following facts. In our calculation we always convert the metric (1) as

$$\mathrm{d}s^2 = C^2(\eta)[\mathrm{d}\eta^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)], \tag{16}$$

where we have introduced the conformal time coordinate η given by

$$\eta = \int^t \frac{\mathrm{d}t}{a(t)}. \tag{17}$$

We then write the equation in η and look for the turning points. To our surprise we find that the turning points occur at $\eta = \pm i\beta$ which is an indication of the fact that this defines the cosmological singularity as in the work of Castagnino and others. This allows us to define the vacuum $t = 0$ i.e., $\eta = -\infty$. The definition of vacuum at $\eta = \infty$ is not a problem. Here the WKB solution exactly coincides with the exact solution of the problem. So the vacuum defined with the WKB solution will be an adiabatic vacuum. It has been shown by Castagnino and others that this type of definition takes into account Hadamard renormalisation [23–26] properly i.e., the Green function shows Hadamard singularity. This discussion allows us to identify the turning points in the complex plane (in most cases Re $t = 0$) as the cosmological singularity though we did not analytically continue the scale factor $a(t)$ for complex t. But in the solution the effect of complex t (paths in imaginary time $\equiv$ paths in euclidean space) is properly taken into account. We now proceed to the calculation of R for various space times.

4. Applications to different space times

We now apply our technique to some known space times to find the pair production amplitude by our method. We consider the motion of a scalar or a Dirac particle in the space time given by (16). For a scalar particle the temporal equation is [18]

$$\ddot{f}_k(\eta) + [k^2 + m^2 C^2(\eta)] f_k(\eta) = 0, \tag{18}$$

and for Dirac particle we have [28, 30]

$$\ddot{f}^{\pm}_{\lambda} + [\lambda^2 + m^2 C^2(\eta) \pm imC(\eta)] f^{\pm}_{\lambda} = 0. \tag{19}$$

where λ is a separation constant. This separation constant arises when one tries to solve the Dirac equation by the method of separation of variables [29] with the substitution

$$\psi_{\lambda jlm} = \exp(-3/2\eta) B_\lambda(\eta) M_{\lambda jl}(r) S(\theta,\phi) Z_{jlm}(\theta,\phi) \tag{20}$$

where

$$0 \leqslant \lambda < \infty, \quad j = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, \quad l = j \pm \tfrac{1}{2}, \quad -j \leqslant m \leqslant j \tag{21}$$

$$B_\lambda(\eta) = \begin{pmatrix} f^+_\lambda(\eta) I & 0 \\ 0 & f^-_\lambda(\eta) I \end{pmatrix}; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{22}$$

and $M_{\lambda jl}(r)$, $S(\theta,\phi)$, $Z_{jlm}(\theta,\phi)$ are defined as in [29]. Substituting (20) and (22) in the Dirac equation one gets two first order equations for $f^{\pm}_{\lambda}$. Equation (19) is obtained from the first order equations (see reference [29]). For a Dirac particle the relation

$$f^+_\lambda(m) = f^-_\lambda(-m), \tag{23}$$

is satisfied. So we consider the solution with one sign only.

We mentioned earlier that the nature of vacuum plays a crucial role in all particle production situations in curved space times. We thus consider two types of problems where in and out regions are static and where either of these two or both are non static.

Case 1. Static in and out regions

We first consider a case where both in and out regions are static. For this purpose we choose a scale factor

$$C^2(\eta) = A + B \tanh \rho\eta. \tag{24}$$

Now we have

$$\begin{aligned} C^2_{\text{in}} &= C^2(\eta \to -\infty) = A - B, \\ C^2_{\text{out}} &= C^2(\eta \to +\infty) = A + B, \end{aligned} \tag{25}$$

so that from (18) we can write

$$\begin{aligned} w^2_{\text{in}} &= k^2 + m^2(A - B) \\ w^2_{\text{out}} &= k^2 + m^2(A + B). \end{aligned} \tag{26}$$

This implies that for both the in and out regions $\partial/\partial\eta$ is the killing vector. As $w_{\text{in}} \neq w_{\text{out}}$ we expect particle production in such a space time.

Equation (18) with (24) is of the form of (2) with

$$\Omega(\eta) = [k^2 + m^2(A + B\tanh\rho\eta)]^{1/2}. \tag{27}$$

The indefinite integral

$$I = \int \Omega(\eta)\mathrm{d}\eta, \tag{28}$$

with the substitution

$$\tau = \exp(\rho\eta) \tag{29}$$

becomes

$$I = \frac{1}{\rho}\int\left(\frac{w_{\text{out}}^2\tau^2 + w_{\text{in}}^2}{\tau^2+1}\right)^{1/2}\frac{\mathrm{d}\tau}{\tau}. \tag{30}$$

Evaluation of (30) gives

$$I = \frac{w_{\text{out}}}{\rho}\ln[(\tau^2+1)^{1/2} + (\tau^2+\gamma^2)^{1/2}] + \frac{w_{\text{out}}\gamma}{2\rho}\ln\left[\frac{(\tau^2+\gamma^2)^{1/2} - \gamma(\tau^2+1)^{1/2}}{(\tau^2+\gamma^2)^{1/2} + \gamma(\tau^2+1)^{1/2}}\right], \tag{31}$$

where we have introduced

$$\gamma = w_{\text{in}}/w_{\text{out}}. \tag{32}$$

With the help of (7) the turning points are determined to be

$$\tau_1 = i\gamma,$$
$$\tau_2 = -i\gamma. \tag{33}$$

Using (5) we evaluate $S(\tau_1\tau_2)$ to be

$$S(\tau_1\tau_2) = 0. \tag{34}$$

We find $S(\tau_1)$ from (6) to be

$$S(\tau_1) = \frac{w_{\text{out}}}{\rho}\ln(1-\gamma^2)^{1/2} + \frac{i}{2\rho}w_{\text{out}}\gamma\pi. \tag{35}$$

From (13), (34) and (35) we evaluate the pair production amplitude to be

$$R = -i\exp\left[\frac{2i}{\rho}w_{\text{out}}\ln(1-\gamma^2)^{1/2}\right]\exp\left(-\frac{\gamma\pi w_{\text{out}}}{\rho}\right), \tag{36}$$

so that we have

$$|R|^2 = \exp\left(-\frac{2\pi w_{\text{in}}}{\rho}\right). \tag{37}$$

The expression (37) has also been calculated using Bogolubov transformation technique [18]. The result is

$$|R|^2 = \frac{|\beta_k|^2}{|\alpha_k|^2} = \frac{\sinh^2 \frac{\pi}{2\rho}(w_{\text{out}} - w_{\text{in}})}{\sinh^2 \frac{\pi}{2\rho}(w_{\text{out}} + w_{\text{in}})} \sim \exp\left(-\frac{2\pi w_{\text{in}}}{\rho}\right), \tag{38}$$

for $w_{\text{in}} < w_{\text{out}}$. This result (38) exactly coincides with (37).

Case 2. Non static in and out region

We now consider space times where the explicit role of conformal and adiabatic vacua has been clarified. We consider two types of universe in 4 dimensions namely the Chitre-Hartle universe

$$ds^2 = dt^2 - t^2(dx^2 + dy^2 + dz^2), \tag{39}$$

and the Milne universe

$$ds^2 = dt^2 - t^2[dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta d\phi^2)] \tag{40}$$

and discuss pair production of spinor particle. For both the cases the metric is given by [29]

$$ds^2 = C^2(\eta)[d\eta^2 - dr^2 - f^2(r)(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{41}$$

where

$$f(r) = r, \sinh r \text{ for } k = 0;\ -1; \quad C(\eta) = \exp \eta. \tag{42}$$

The vacuum expectation value of the regularised energy momentum tensor for spinor fields has been evaluated [29] in the background of the Chitre-Hartle and Milne metric in the framework of the one-loop approximation to quantum gravity. As in the case of scalar particles, it has been shown that

$$\langle T^{(1/2)}_{ik} \rangle^{\text{Milne}}_{\text{reg}} = \langle T^{(1/2)}_{ik} \rangle^{\text{Chitre}}_{\text{reg}} = 0. \tag{43}$$

Another method to foresee particle production is the non perturbative technique of complex multiple reflection in time. We verify with our method whether or not there is particle production in adiabatic vacuum. The one loop correction is basically an approximation where $O(\hbar)$ correction is taken into account. But in our method we have taken corrections non perturbatively as $O(\hbar) + O(\hbar)^2 + \ldots$ In this sense our method will be checked in the light of the correctness of one loop approximation as well as renormalization procedure.

Equation (19) with (42) reads

$$\ddot{f}_\lambda + [\lambda^2 + m^2 \exp(2\eta) + im \exp(\eta)] f_\lambda = 0 \tag{44}$$

Equation (44) is of the form of (2) with

$$\Omega(\tau) = (\lambda^2 + i\tau + \tau^2)^{1/2}, \tag{45}$$

where we have substituted

$$\tau = m \exp(\eta). \tag{46}$$

The turning points are determined from (7) to be

$$\tau_1 = -i/2 + i(\lambda^2 + \tfrac{1}{4})^{1/2},$$
$$\tau_2 = -i/2 - i(\lambda^2 + \tfrac{1}{4})^{1/2}. \tag{47}$$

Using (5) we calculate $S(\tau_1 \tau_2)$ to be

$$S(\tau_1 \tau_2) = \pi/2 + i\pi\lambda. \tag{48}$$

Using (6) we find $S(\tau_1)$ to be

$$S(\tau_1) = -\pi/4 + i[\ln(\lambda^2 + \tfrac{1}{4}) + \pi\lambda/2]. \tag{49}$$

The pair production amplitude can now be obtained from (13), (48) and (49) to be

$$R = -i\frac{1}{(\lambda^2 + \frac{1}{4})^{1/2}} \cdot \frac{\exp(-i\pi/2)\exp(-\pi\lambda)}{1 - \exp(-2\pi\lambda)}, \tag{50}$$

so that we have

$$|R|^2 = \frac{1}{(\lambda^2 + \frac{1}{4})} \cdot \frac{\exp(-2\pi\lambda)}{[1 - \exp(-2\pi\lambda)]^2}. \tag{51}$$

The consequence of this result will be discussed shortly.

Case 3. Static in and non-static out region

We consider now the scale factor with the past statically bounded expansion,

$$C(\eta) = C_{\text{in}} + \exp(2b\eta), \tag{52}$$

where $b = \text{constant} > 0$ is the expansion parameter. The expansion starts at $\eta \to -\infty$ with the constant scale factor $C_{\text{in}} = \text{constant}$ which we assume to be always $C_{\text{in}} \neq 0$; thus the in region is Minkowskian.

Equation (19) with (52) becomes

$$\ddot{f}_\lambda + [w_{\text{in}}^2 + 2m(mC_{\text{in}} + ib)\exp(2b\eta) + m^2\exp(4b\eta)]f_\lambda = 0 \tag{53}$$

where we have put

$$w_{\text{in}}^2 = \lambda^2 + m^2 C_{\text{in}}^2. \tag{54}$$

Equation (53) is of the form of (2) with

$$\Omega(\tau) = [\tau^2 + (2\mu + i)\tau + \nu^2]^{1/2}, \tag{55}$$

where we have introduced

$$\tau = \frac{m}{2b}\exp(2b\eta), \tag{56}$$

$$\nu = w_{\text{in}}/2b, \tag{57}$$

$$\mu = mC_{\text{in}}/2b. \tag{58}$$

The turning points are determined from (7) to be

$$\tau_1 = -(\mu + i/2) + [(\mu + i/2)^2 - \nu^2]^{1/2},$$
$$\tau_2 = -(\mu + i/2) - [(\mu + i/2)^2 - \nu^2]^{1/2}. \tag{59}$$

Using (5) we calculate $S(\tau_1 \tau_2)$ to be

$$S(\tau_1 \tau_2) = \pi/2 + i(\nu - \mu)\pi. \tag{60}$$

Using (6) we have for $S(\tau_1)$

$$S(\tau_1) = \mathrm{Re} + i[\ln\{(\mu^2 - \tfrac{1}{4} - \nu^2)^2 + \mu^2\}^{1/8} + \tfrac{1}{2}\mu \tan^{-1}\frac{\mu}{\mu^2 - \frac{1}{4} - \nu^2} + \frac{\nu\pi}{2}] \tag{61}$$

where 'Re' refers to the real part of $S(\tau_1)$. From (13), (60) and (61) we have for the pair production amplitude

$$R = -i[(\mu^2 - \tfrac{1}{4} - \nu^2)^2 + \mu^2]^{1/4} \times \frac{\exp(2i\,\mathrm{Re})\exp\left(-\mu\tan^{-1}\frac{\mu}{\mu^2 - \frac{1}{4} - \nu^2}\right)\exp(-\nu\pi)}{1 - \exp[-i(\nu - \mu)\pi]} \tag{62}$$

so that we have

$$|R|^2 = \frac{1}{[(\mu^2 - \frac{1}{4} - \nu^2)^2 + \mu^2]^{1/2}} \frac{\exp\left[-\mu\tan^{-1}\frac{\mu}{\mu^2 - \frac{1}{4} - \nu^2}\right]\exp(-2\nu\pi)}{[1 - \exp\{-2(\nu - \mu)\pi\}]^2} \tag{63}$$

In case $C_{\mathrm{in}} = 0$ we have $\mu = 0$ and (63) reduces to

$$|R|^2 = \frac{1}{(\nu^2 + \frac{1}{4})} \cdot \frac{\exp(-2\nu\pi)}{[1 - \exp(-2\nu\pi)]^2}, \tag{64}$$

which is the same result (51) we obtained from Chitre-Hartle and Milne metric.

We now discuss the results we have so far obtained. For the static in and out regions [case 1] our results exactly reproduce the result of Birrel and Davies [18].

In the case of Chitre-Hartle and Milne universe [case 2] our result needs a careful analysis. It has been mentioned that there will be no particle production in Chitre-Hartle or Milne case in adiabatic vacuum though we see pair production in conformal vacuum. But in our analysis through WKB approximation the vacuum chosen is definitely adiabatic vacuum. If we translate our results to two dimensions, we find particle production even in adiabatic vacuum though this is not actually the case as demanded by Sahni [29]. Our suggestion is that this type of reduction should not be done directly from the results of $|R|^2$ in 4 dimensions. The reason is as follows. Elsewhere [9, 21] we calculated $|R|^2$ in degenerate Kasner's space time, namely

$$ds^2 = dt^2 - dx^2 - t^2 dy^2 - dz^2. \tag{65}$$

We found a non-zero $|R|^2$ for such a space time. Using $x=0$, $z=0$, thereby setting $k_1=k_3=0$ in the expression for $|R|^2_{\text{kasner's space time}}$, we found $|R|^2 \longrightarrow 0$ i.e., $|R|^2_{\text{Milne universe}} \longrightarrow 0$. This is quite an usual result. The reason is that the 2 dimensional Milne universe can be reduced to a flat space time under the substitution [18]

$$y^0 = \exp\eta \cosh x$$
$$y^1 = \exp\eta \sinh x \tag{66}$$

with $0 < y^0 < \infty$ and $-\infty < y^1 < \infty$. Hence we do not expect particle production in 2 dimensional Milne universe. But the 4 dimensional Milne universe is not 'merely flat space time in disguise'. Naturally in our calculation [9] we find that $|R|^2_{\text{Milne in 4 dimension}} \neq 0$. As λ in (51) does not explicitly contain k_1, k_2, k_3 it is difficult to set any two of the three variables to be equal to zero. In other words, a direct reduction of λ(4 dimension) to λ(2 dimension) is not readily possible. Hence passage from 4 dimensions to 2 dimensions is not justified. A definite conclusion can be obtained by considering the Dirac equation in 2 dimensional Milne universe and investigating particle production aspect in such a space time. But Milne universe in 4 dimension is not directly related to the Minkowski covering space to give a special significance to adiabatic vacuum. In this respect non-zero $|R|^2$ in adiabatic vacuum for Milne universe in 4 dimension is not unusual. However, the standard result that there will be no particle production in adiabatic vacuum follows from our expression of $|R|^2$. If the ψ_{WKB} is to be identified with the adiabatic vacuum then in the expression (13) we should put m or λ large in $\exp[\pm iS(t)]$. For large λ the vacuum defined by (19) will then reproduce the results of adiabatic vacuum. In that case from (51) we have $|R|^2 \longrightarrow 0$, because of the exponential factor. The surprising fact is that (51) is independent of m indicating that the vacuum in question does not depend upon m. It should be noted that adiabatic vacuum is a result of large mass approximation whereas the conformal vacuum exploits the conformal symmetry of the massless case. Our non-perturbative calculation takes the contribution from disconnected region and origin of the thermal spectrum [denominator of (51)] is inherent in our approach. In (51) term $\sim 1/\lambda^2$ generally arises in standard calculations but the rest is due to the contribution of tunnelling behaviour in complex treatment. The origin of the thermal spectrum is basically due to the contribution from the bounce points which depends on the topology of the space time chosen. It is evident that if we consider the adiabatic vacuum we must take $\lambda \longrightarrow$ large in the expression for $|R|^2$ but if the vacuum is not adiabatic (of course our vacuum is not conformal) then the approximation λ large is not necessary and the origin of thermal spectrum is definitely related to the topology of the spacetime. This requires further investigation.

Lotze [19] has also investigated the statically bounded expansion cases [case 3]. However our result is slightly different from his result. Let us suppose that the expansion is rapid. By this we mean that the expansion parameter b is so large that at Compton time $t_c = 1/m$, C_{in} may be omitted in $C(\eta)$. In that case we have $a(t_c)$ [see (1)] approximately equal to $2bt_c > C_{\text{in}}$ and hence $\mu_{\text{in}} \ll 1$, a condition of rapid expansion. In expanding universe λ/a is the momentum where λ is a separation constant in (19). So the non-relativistic particles satisfy the condition $\lambda/am \ll 1$, i.e., $\lambda/(2bt_c m) \ll 1$, which means $\lambda/2b \ll 1$ (since $mt_c = 1$). Thus even in the expanding universe the particles can be non-relativistic so long as b is large i.e., expansion is rapid. We finally conclude that the vacuum defined in Chitre-Hartle or Milne case is not exactly the adiabatic vacuum though it behaves as adiabatic vacuum only in large λ approximation. This is an interesting result. However it is a matter of investigation to judge this vacuum with respect to strong and weak vacuum as defined by Castagnino and others [23–26]. At least we can say that the vacuum chosen to WKB approximation in complex time is not adiabatic vacuum for λ, m not large, even if the conditions (15) are satisfied.

5. Conclusion

Let us summarize the essential features and the conclusions that follow therefrom. (1) We employ WKB approximation in complex time to find out pair production amplitude. Real path corresponds to the standard WKB approximation but complex semiclassical paths in time takes into account the tunnelling behaviour. (2) It is a non-perturbative approach, at least a step further from the one-loop approximation. (3) The use of WKB method in this type of problem is quite justified. (4) Thermal spectrum arises from the tunnelling path. Classically, the particle is at rest at the turning points but the quantum behaviour makes the particle to tunnel into complex plane to come back as if it spends most of the time at the turning points. (5) The method is applied to various cases almost reproducing the exact result obtained through Bogolubov transformation technique. (6) It is a noticeable result that the turning points in most cases studied lies at $t = 0 \pm i\varepsilon$. This suggests that particle production takes place at very early stage of the universe. Behaviour of the particle in the euclidean space determines the nature of production. This aspect is very important in the light of quantum cosmology particularly in evaluating quantum wavefunction. (7) We have chosen our vacuum such that the scale factor satisfies the criteria laid down by Castagnino and others [23–26] i.e., we are dealing with adiabatic vacuum. (8) We have also evaluated the amount of particle production in a de-Sitter space time with $a(t) = \exp(2bt)$ and found a pair production $\alpha \dfrac{\exp(-m\pi/b)}{1 + \exp(-m\pi/b)}$. This is due to the fact that the adiabatic vacuum reduces to conformal vacuum. The particle production in de-Sitter space time, stability of de-Sitter vacuum and evolution of cosmological constant will be dealt with in future.

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