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A note on the harmonic plus inverse-harmonic potential in quantum mechanics

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Abstract. The ground state of a class of potentials described by $V(x) = a_{20}x^2 + a_1/x^2$ is investigated. This potential possesses some distinct features like that of a symmetric double-well potential, and can be used in modelling the fusion process of two identical nuclei as well.

Keywords. Schrödinger equation; eigenfunction; square integrability.

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It is well-known [1] that as a result of the separation of the Schrödinger equation in radial and angular coordinates, a central potential $v(r)$ attains an effective form $\mathscr{V}(r) = v(r) + l(l+1)\hbar^2/(2\mu r^2)$ when appears in the reduced Schrödinger equation. Here the integer values of l (called the orbital quantum number) correspond to the orbital excitations of the system described by $v(r)$. In this note we study the one-dimensional system described by the potential

$$V(x) = a_{20}x^2 + a_1/x^2. \tag{1}$$

It is noticed that the square integrability as well as the single-valued nature of the eigenfunctions for (1) in the region $x \in (-\infty, \infty)$ demand some discrete values of a_1 (namely, $a_1 = m(m-1)\hbar^2/2\mu$, where m is an integer which is allowed to take the values $m = 2, 3, 4, \ldots$, and it should not be confused with the standard magnetic quantum number). As a result there appears a striking similarity between the three-dimensional central potential $\mathscr{V}(r)$ for the harmonic form of $v(r)$ and the one-dimensional potential (1). While it may be interesting to note this type of similarity, a difference worth mentioning is that the discrete values of l describe the excitations of the same system whereas the discrete values of a_1, correspond altogether to different systems. Moreover, the origins of $l(l+1)$-term in $\mathscr{V}(r)$ and that of $m(m-1)$-term in $V(x)$ are different. The study of such distinct features for the potential (1) is new.

The Schrödinger equation for the potential (1) can be written as

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b_{20}x^2 + \frac{b_1}{x^2}\right)\psi(x) = \lambda\psi(x), \tag{2}$$

where $b_{20} = 2\mu a_{20}/\hbar^2$, $b_1 = 2\mu a_1/\hbar^2$ and $\lambda = 2\mu E/\hbar^2$. In order to solve (2) we make use [2] of an ansatz for the eigenfunction, namely $\psi(x) = N\exp((g(x)))$, with $g(x) = \beta_{20}x^2 + \beta_1 \ln x$. Using this form in (2) and subsequently rationalizing the resultant equation, one obtains the unknown constants β_{20} and β_1 in terms of b_{20} and b_1 as

$$\beta_{20} = \pm\sqrt{b_{20}}/2; \quad \beta_1 = [1 \pm (1+4b_1)^{1/2}]/2.$$

However, for the well behaved nature of the solution $\psi(x)$ at $x = \pm\infty$, one should choose the negative sign in β_{20} and the positive sign in β_1. This will yield the expressions for the eigenvalue λ and the eigenfunction $\psi(x)$ as

$$\lambda = [2 + (1+4b_1)^{1/2}]\sqrt{b_{20}}, \tag{3a}$$

$$\psi(x) = N x^{1/2\{1+(1+4b_1)^{1/2}\}} \exp(-\sqrt{b_{20}}x^2/2), \tag{3b}$$

where the normalization constant, N, can be determined from

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x = 1. \tag{4}$$

It may be noted that for the existence of integral (4) the exponent of x in (3b) should be a positive integer including zero (say m) i.e.

$$1 + (1+4b_1)^{1/2} = 2m, \quad (m = 0, 1, 2, \ldots)$$

implying

$$b_1 = m(m-1). \tag{5}$$

In fact for $m = 0$ and 1, b_1 becomes zero and thus reducing the problem to the pure harmonic case. Further, for any nonzero contribution from the inverse-harmonic term in (1) one should have $m \geqslant 2$. Finally, λ and $\psi(x)$ for the potentials

$$V(x) = a_{20}x^2 + \frac{m(m-1)\hbar^2}{2\mu x^2}, \quad (m = 2, 3, 4, \ldots) \tag{6}$$

turn out to be

$$\lambda = (2m+1)\sqrt{b_{20}}, \tag{7a}$$

$$\psi(x) = N x^m \cdot \exp(-\sqrt{b_{20}}x^2/2), \tag{7b}$$

with the normalization constant given by

$$N = \left[\frac{2^m b_{20}^{1/2(m+1/2)}}{\sqrt{\pi}(2m-1)!!}\right]^{1/2}.$$

For a typical value of b_{20} (say $b_{20} = 2{\cdot}5$) the behaviour of the potential (6) and that of the corresponding eigenfunctions is shown in figures 1 and 2, respectively for $m = 2$, 4 and 7. Here we have set $\hbar = 2\mu = 1$. For $m \geqslant 2$, figure 1 clearly shows the features of a symmetric double-well potential however with an infinite barrier between the two wells. Further, a uniformly shifting tendency of the minima with respect to the increasing values of m can be noticed alongwith the fact that the energy levels are identical in both the wells. While for even values of m, $\psi(x)$ has two maxima (cf. figure 2) lying symmetrically on either side at $x = \pm[m(m-1)/b_{20}]^{1/4}$, there exists a node at $x = 0$ in $\psi(x)$ for odd values of m.

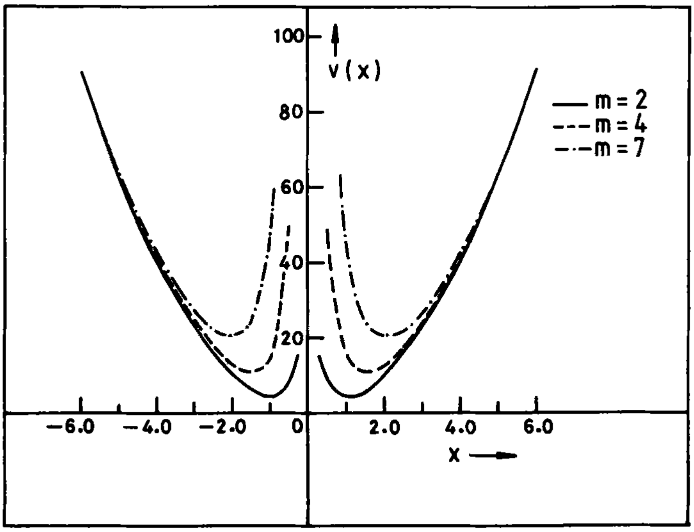

Figure 1. Plot of potential (6) for different values of m and for $\hbar = 2\mu = 1$; $b_{20} = 2{\cdot}5$.

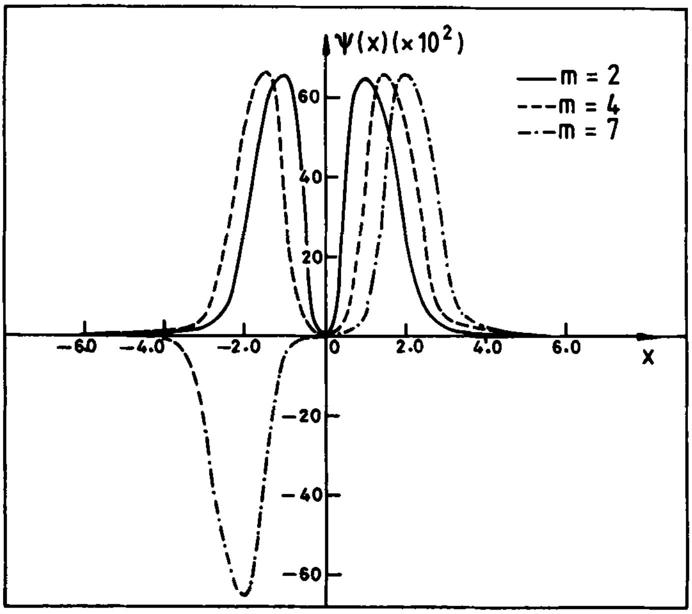

Figure 2. Plot of the ground state normalized wave functions for parameter values defined for figure 1.

From the point of view of applications of the potential (1) a few remarks are in order: (i) Here we have considered only the ground state of (1). For the excited states, however, one can as well start with the ansatz $\psi(x) = Nf(x)\cdot\exp(g(x))$, where $f(x)$ is a polynomial and $g(x)$ can be chosen as before, and obtain [3] the solution of (2); or else one can look for the solutions of (2) directly in terms of the Laguerre polynomials. (ii) Generalization of (1) to the noncentral case in two-dimensions is also possible [2] in a straightforward manner, e.g. the cases (a) and (b) of § 2 (cf. [2]) are worth investigating in this context. (iii) Potential (1) can offer a model-description for the fusion of two identical nuclei, particularly when each nucleus is capable of being described in terms of a harmonic potential. In fact, by reducing successively the values of b_1 (or that of m) and finally choosing $b_1 = 0$, the two nuclei can be put together in the same harmonic well. In other words, the integer values of m accounts for the relative separation between the two wells (cf. figure 1) which are expected to have the identical energy levels. As a matter of fact such an idealised description will overcome some of the difficulties which already arise in the conventional two-centre model for nuclei. For example, the problem of eliminating the barrier between the two wells or that of the corrections needed [4] for the quantum numbers of the one well in the presence of the other, is automatically resolved in the present description. This description, however, remains more of an illustrative type than the realistic one in the sense that the problem is solved only in one dimension. (iv) Potential (1) can also be useful in the description of those field theories or the structural phase transitions (see, for example, Khare and Behera [5]) in which the role of tunnelling is either of least importance or not at all important. It is also possible to study the phenomenon of tunnelling by introducing some perturbation in the potential (1) and consequently by generating an asymmetry between the two wells. In this case, however, the ground state (7b) can be used to develop the higher order perturbation solutions for different values of m.

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References

[1] See, for example, L I Schiff, *Quantum mechanics* (McGraw Hill Book Co. Inc. 1955) p. 74; or S Gasiorowicz, *Quantum physics* (John Wiley and Sons, 1974) p. 179
[2] R S Kaushal, *Ann. Phys.* **206**, 90 (1991)
[3] R S Kaushal and D Parashar, *Phys. Lett.* **A170** 335 (1992)
D R Taylor and P G L Leach, *J. Math. Phys.* **30**, 1525 (1989)
[4] P -T Ong and W Scheid, *Z. Naturforsch.* **30a**, 406 (1975)
[5] A Khare and S N Behera, *Pramana – J. Phys.* **14**, 327 (1980)