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Coherent states and squeezed states of real q-deformed quantum oscillators

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Abstract. A detailed physical characterisation of the coherent states and squeezed states of a real q-deformed oscillator is attempted. The squeezing and q-squeezing behaviours are illustrated by three different model Hamiltonians, namely i) Batemann Hamiltonian ii) harmonic oscillator with time dependent mass and frequency and iii) a system with constant mass and time-dependent frequency.

Keywords. Quantum groups; quantum oscillators; q-coherent states; q-squeezed states.

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1. Introduction

Quantum groups and quantum oscillators have been widely studied recently. Biedenharn [1] and Macfarlane [2] discussed q-oscillators and realized the quantum algebra $su_q(2)$ in terms of these oscillators. Many other versions of q-oscillator have appeared [3, 4]. Quantum oscillators have already found applications in diverse fields such as molecular spectroscopy [5], condensed matter physics [6], quantum optics [4, 7] and many-body theory [8]. An anharmonic version of q-oscillator with quartic interaction has been discussed recently [9].

Coherent states and squeezed states of the electromagnetic field have been the subject of several investigations. Schrodinger [10] introduced a system of wavefunctions to describe the nonspreading wave packets for the quantum harmonic oscillator. Later Glauber [11] called these states coherent states (CS) and applied them to the radiation field. Squeezed states were first introduced by Yuen [12] and are of great importance in quantum optics. Coherent states and squeezed states can be defined for the q-oscillators also and are of current interest [4, 7]. Chaturvedi *et al* [4] have constructed the coherent states and squeezed states corresponding to the commutation relation $aa^+ - qa^+a = 1$. In this paper we compute the variances associated with the coherent and squeezed states and attempt to distinguish between squeezing and q-squeezing for real q-deformed oscillators.

Section 2 is a brief review of the spectrum of a real q-deformed oscillator. In § 3 a general treatment of the coherent states and squeezed states is given, emphasising the physical aspects. Section 4 deals with these model Hamiltonians that illustrate the ideas and formalism of squeezing and q-squeezing under the SU(1, 1) pattern. The results of numerical computations of the probability for observing a definite number of quanta in q-squeezed states are presented and some concluding remarks are offered in § 5.

2. Spectrum of a q-deformed oscillator

Here we recapitulate the basic facts about real q-deformed oscillators [4]. Let us consider the q-commutation relation (q-CR)

$$aa^{+} - qa^{+}a = 1 \tag{1}$$

where q is real. We demand that

$$\begin{aligned}[N, a^{+}] &= a^{+}, \\ [N, a] &= -a.\end{aligned} \tag{2}$$

If we take

$$\begin{aligned}aa^{+} &= [N+1]_q \\ a^{+}a &= [N]_q\end{aligned} \tag{3}$$

with $[X]_q = (q^X - 1)/(q-1)$, (1) is readily satisfied. With the help of (3), (1) can be rewritten as

$$[a, a^{+}] = q^N. \tag{4}$$

The operators a^{+}, a and N can be thought of as creation, annihilation and number operators respectively for a q-deformed H.O. with the Hamiltonian

$$H_q = P_q^2/2m + \tfrac{1}{2}m\omega^2 X_q^2 \tag{5}$$

where

$$\begin{aligned}X_q &= \sqrt{\hbar/2m\omega}(a + a^{+}), \\ P_q &= -i\sqrt{\frac{m\hbar\omega}{2}}(a - a^{+}),\end{aligned} \tag{6}$$

X_q and P_q are hermitian operators satisfying the CR

$$[X_q, P_q] = i\hbar q^N. \tag{7}$$

The uncertainty relation corresponding to this is

$$\Delta X_q \Delta P_q \geqslant \frac{\hbar}{2}\langle q^N \rangle.$$

As $q \to 1$, one recovers the corresponding operators of ordinary quantum mechanics namely X and P.

The q-oscillator Hamiltonian can be written as

$$\begin{aligned}H_q &= \frac{\hbar\omega}{2}(aa^{+} + a^{+}a), \\ &= \frac{\hbar\omega}{2}([N]_q + [N+1]_q).\end{aligned} \tag{8}$$

The energy eigenvalues are

$$E_n = \tfrac{1}{2}\hbar\omega([n]_q + [n+1]_q) \tag{9}$$

where n is an eigenvalue of the operator N:

$$N|n\rangle_q = n|n\rangle_q. \tag{10}$$

In general, any state $|n\rangle_q$ can be built up from the q-vacuum

$$|n\rangle_q = \frac{(a^+)^n}{\sqrt{[n]!}}|0\rangle_q \tag{11}$$

where $[n]! = [n]\,[n-1]\cdots[2][1]$ and the q-vacuum $|0\rangle_q$ is identified with ordinary vacuum $|0\rangle$.

3. Coherent states and squeezed states

A normalized q-coherent state (q-CS) is defined as

$$a|\alpha\rangle_q = \alpha|\alpha\rangle_q \tag{12}$$

$$|\alpha\rangle_q = [\exp_q(|\alpha|^2)]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}}|n\rangle_q \tag{13}$$

where

$$\exp_q\chi = \sum_{n=0}^{\infty} \frac{\chi^n}{[n]!}$$

The q-CS can be obtained from the q-vacuum by the action of a q-displacement operator $D_q(\alpha)$:

$$|\alpha\rangle_q = D_q(\alpha)|0\rangle_q \tag{14}$$

where

$$D_q(\alpha) = [\exp_q|\alpha|^2]^{-1/2}\exp_q(\alpha a^+).$$

Note that the displacement operator that generates a q-CS from the vacuum is not equal to $\exp_q(\alpha a^+ - \alpha^* a)$, which is the q analogue of $\exp(\alpha a^+ - \alpha^* a)$, the displacement operator in the $q = 1$ theory.

The probability of measuring n quanta in the q-CS is

$$\begin{aligned} P_n &= [\exp_q(|\alpha|^2)]^{-1}\left(\frac{|\alpha|^{2n}}{[n]!}\right) \\ &= [\exp_q(\langle[n]\rangle)]^{-1}\left(\frac{\langle[n]\rangle^n}{[n]!}\right) \end{aligned} \tag{15}$$

where we have made use of the fact that

$$|\alpha|^2 = \langle[n]\rangle. \tag{16}$$

Equation (15) represents a q-Poisson distribution. Variances of X_q and P_q defined by (7) are calculated:

$$\mathrm{Var}\, X_q = \frac{\hbar}{2m\omega}(|\alpha|^2(q-1)+1),$$

$$\mathrm{Var}\, P_q = \frac{m\hbar\omega}{2}(|\alpha|^2(q-1)+1). \tag{17}$$

Equation (17) fixes an upper limit for α i.e. $|\alpha| \leqslant 1/(1-q)$.

The uncertainty product for a q-CS$|\alpha\rangle_q$ is expressed in the form

$$(\Delta X_q)(\Delta P_q) = \hbar/2(|\alpha|^2(q-1)+1) \tag{18}$$

while the corresponding relation for the state $|n\rangle_q$ is

$$(\Delta X_q)_n(\Delta P_q)_n = (\hbar/2)([n]+[n+1]). \tag{19}$$

The q-vacuum state satisfies the relation

$$(\Delta X_q)_0(\Delta P_q)_0 = \hbar/2 \tag{20}$$

which coincides with the position-momentum uncertainty relation for the ordinary vacuum state $|0\rangle$. Thus the uncertainty product for q-CS is different from that of the q-vacuum state, which is $\hbar/2$. Also the uncertainty product depends on the parameter α.

Let us introduce two self-adjoint operators A_1 and A_2, called the quadrature components, such that

$$a = A_1 + iA_2$$

$$a^+ = A_1 - iA_2 \tag{21}$$

then it can be seen that

$$\mathrm{Var}\, A_1 = (\tfrac{1}{4})((q-1)|\alpha|^2+1),$$

$$\mathrm{Var}\, A_2 = (\tfrac{1}{4})((q-1)|\alpha|^2+1),$$

$$(\Delta A_1)(\Delta A_2) = \tfrac{1}{4}((q-1)|\alpha|^2+1). \tag{22}$$

The quantum noise energy [12] is zero for q-CS also:

$$\langle\alpha|(\Delta a)^+(\Delta a)|\alpha\rangle_q = 0. \tag{23}$$

Let us now define two new operators b and b^+:

$$\begin{pmatrix} b \\ b^+ \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \nu^* & \mu^* \end{pmatrix}\begin{pmatrix} a \\ a^+ \end{pmatrix}. \tag{24}$$

With

$$|\mu|^2 - |\nu|^2 = 1 \tag{25}$$

it follows that

$$bb^+ - b^+b = q^N. \tag{26}$$

The SU(1, 1) squeezed states of the q-oscillator are defined as follows:

$$b|\beta\rangle_q = \beta|\beta\rangle_q. \tag{27}$$

We can express $|\beta\rangle_q$ as a linear combination of the number states $|n\rangle_q$ [13]:

$$|\beta\rangle_q = \sum_{n=0}^{\infty} C_n|n\rangle_q, \tag{28}$$

$$b|\beta\rangle_q = (\mu a + v a^+) \sum_{n=0}^{\infty} C_n|n\rangle_q,$$

$$= \beta \sum_{n=0}^{\infty} C_n|n\rangle_q. \tag{29}$$

It follows that

$$C_1 = \beta C_0/\mu$$

$$C_2 = (\beta C_1 - v C_0)/\sqrt{([2])}\mu$$

or in general,

$$C_n = \frac{\beta C_{n-1} - v\sqrt{([n-1])}C_{n-2}}{\mu\sqrt{[n]}}. \tag{30}$$

Although, in general μ and v are arbitrary, here we consider the evolution of a state which is initially a q-coherent state (i.e. $\mu(0) = 1$, $v(0) = 0$) into a q-squeezed state $|\beta\rangle_q$ at t. This procedure is motivated by similar considerations made in references [14] and [15] in the context of ordinary coherent states and squeezed states.
Inverting (24), we have

$$a = \mu^* b - v b^+$$

$$a^+ = v^* b + \mu b^+$$

so that the Hamiltonian (8) becomes

$$H = \frac{\hbar\omega}{2}(2\mu^* v^* b^2 - 2\mu v b^{+2} + bb^+ + b^+ b). \tag{31}$$

It can easily be shown that

$$\langle A_1 \rangle_\beta = 1/2(\beta(\mu^* - v^*) + \beta^*(\mu - v)), \tag{32}$$

$$\langle A_2 \rangle_\beta = 1/2(\beta^*(\mu + v) + \beta(\mu^* + v^*)),$$

$$\mathrm{Var}\, A_1 = \tfrac{1}{4}|\mu - v|^2 \sigma(t),$$

$$\mathrm{Var}\, A_2 = \tfrac{1}{4}|\mu + v|^2 \sigma(t). \tag{33}$$

Also $(\Delta A_1)(\Delta A_2) = \tfrac{1}{4}|\mu^2 - v^2|\sigma(t)$ (34)
where

$$\sigma(t) = \sum_{n=0}^{\infty} q^n |C_n|^2. \tag{35}$$

In the present work we restrict q to the domain $0 \leqslant q \leqslant 1$, and assume the normalization

$$\sum_{n=0}^{\infty} |C_n|^2 = 1. \tag{36}$$

This yields the inequality $\sigma(t) \leqslant 1$.

At this point we wish to draw a distinction between squeezing and q-squeezing. By squeezing we mean that the variance goes below the value of the uncertainty product in the q-vacuum state, which is same as that for the ordinary state. On the other hand, q-squeezing implies that the variance is less than the uncertainty product for the q-CS state. For an undeformed oscillator the uncertainty product is the same for vacuum as well as for CS. But when $q \neq 1$, the uncertainty product has different values corresponding to q-vacuum and q-CS.

From (35),

$$\sigma(0) = \frac{\exp_q(q|\beta|^2)}{\exp_q(|\beta|^2)}. \tag{37}$$

We assume that $|\beta\rangle_q$ is prepared initially as a q-CS. Hence it follows from (22), (34) and (37) that

$$(q-1)|\alpha|^2 + 1 = \frac{\exp_q(q|\beta|^2)}{\exp_q(|\beta|^2)} \tag{38}$$

Thus in the $q \neq 1$ theory, α and β are interrelated. Squeezing means (ΔA_1) or $(\Delta A_2) \leqslant \frac{1}{4}$ and q-squeezing means (ΔA_1) or $(\Delta A_2) \leqslant \sigma(0)/4$. We may define the degree of squeezing and the degree of q-squeezing respectively as

$$S^{(A)} = \frac{2\langle(\Delta A)^2\rangle - \frac{1}{4}}{\frac{1}{4}}, \tag{39}$$

$$S_q^{(A)} = \frac{2\langle(\Delta A)^2\rangle - (\frac{1}{4})\sigma(0)}{\frac{1}{4}\sigma(0)}. \tag{40}$$

Squeezing corresponds to $S^{A_1} < 0$, or $S^{A_2} < 0$, while q-squeezing implies $S_q^{A_1} < 0$ or $S_q^{A_2} < 0$.

4. SU(1, 1) squeezed states of real q-deformed Hamiltonians

4.1 *Batemann Hamiltonian*

Baseia *et al* [14] studied the appearance of squeezed states for the Batemann Hamiltonian where the mass of the oscillator changes suddenly. In this section we discuss the squeezing and q-squeezing properties associated with SU(1, 1) squeezed states of a real q-deformed Batemann Hamiltonian, defined by the relation

$$H(t) = \frac{P^2}{2M(t)} + \frac{1}{2}M(t)\omega^2 X^2 \tag{41}$$

where

$$M(t) = M_0 e^{\lambda t}.$$

We take the corresponding q-deformed Hamiltonian in the form:

$$H_q(t) = \frac{P_q^2}{2M(t)} + \frac{1}{2}M(t)\omega^2 X_q^2 \tag{42}$$

Setting

$$\mu(t) = \cosh(\lambda t/2) = \frac{1}{2}\left(\sqrt{\frac{M(t)}{M_0}} + \sqrt{\frac{M_0}{M(t)}}\right)$$

$$\nu(t) = \sinh(\lambda t/2) = \frac{1}{2}\left(\sqrt{\frac{M(t)}{M_0}} - \sqrt{\frac{M_0}{M(t)}}\right), \tag{43}$$

the condition $|\mu|^2 - |\nu|^2 = 1$ is easily satisfied. Now consider the case where mass changes suddenly as

$$M_0 \to M_1 \text{ at } t = t_1:$$

$$M(t) = M_0\theta(t - t) + M_1\theta(t - t_1), \tag{44}$$

where θ is the Heaviside step-function.

The quadrature components A_1 and A_2 have equal variances:

$$(\Delta A_1)_\beta^2 = (\Delta A_2)_\beta^2 = \tfrac{1}{4}\sigma(t) = (\Delta A_1)_\beta(\Delta A_2)_\beta. \tag{45}$$

Thus $|\beta\rangle_q$ behaves as a q-CS.

For $t > t_1$,

$$M(t) = M_0 + \Delta M = M_0(1 + \delta)$$

where

$$\delta = \frac{\Delta M}{M_0}, \quad \Delta M = M_1 - M_0$$

Then

$$(\Delta A_1)^2 = \frac{\sigma(t)}{4(1+\delta)},$$

$$(\Delta A_2)^2 = \left(\frac{1+\delta}{4}\right)\sigma(t),$$

$$(\Delta A_1) = (\Delta A_2) = \frac{\sigma(t)}{4}. \tag{46}$$

This implies that the uncertainty product can go below $\frac{1}{4}$. For $(\sigma(t) - 1) < \delta < (1/[\sigma(t)] - 1)$, both $(\Delta A_1)^2$ and $(\Delta A_2)^2$ fall below this value. In this region one can in principle measure both A_1 and A_2 with uncertainties less than that predicted by Heisenberg's uncertainty principle. Also in the region $([\sigma(t)]/[\sigma(0)] - 1) < \delta < ([\sigma(0)]/[\sigma(t)] - 1)$, both A_1 and A_2 have variances below that in q-CS. The squeezing pattern is as follows:

$$A_1 \text{ is squeezed if } \delta > (\sigma(t) - 1),$$

$$A_1 \text{ is } q\text{-squeezed if } \delta > (\sigma(t)/\sigma(0) - 1). \tag{47}$$

Similar remarks apply to squeezing and q-squeezing of the A_2 component.

4.2 *Harmonic oscillator with time dependent mass and frequency*

Consider the Hamiltonian

$$H = P^2/2M(t) + (1/2)M(t)\omega^2(t)X^2$$

where

$$M(t) = M_0 e^{\lambda t}, \quad \omega(t) = \omega_0 e^{-\rho t}. \tag{48}$$

Squeezing properties of this Hamiltonian have been studied earlier [15]. We can q-deform it as

$$H_q = \frac{P_q^2}{2M(t)} + (1/2)M(t)\omega^2(t)X_q^2. \tag{49}$$

The coefficients $\mu(t)$ and $\nu(t)$ are expressed as

$$\mu(t) = \cosh\left(\frac{\lambda-\rho}{2}\right)t,$$

$$\nu(t) = \sinh\left(\frac{\lambda-\rho}{2}\right)t. \tag{50}$$

The variances of the quadrature components are calculated as:

$$(\Delta A_1)^2 = \tfrac{1}{4}e^{-(\lambda-\rho)t}\sigma(t),$$

$$(\Delta A_2)^2 = \tfrac{1}{4}e^{(\lambda-\rho)t}\sigma(t),$$

$$(\Delta A_1)(\Delta A_2) = \tfrac{1}{4}\sigma(t). \tag{51}$$

Here also the uncertainty product may decrease below $\frac{1}{4}$.

A_1 is squeezed if $(\lambda-\rho) < \frac{-1}{t}\ln(\sigma(t))$ and q squeezed if $(\lambda-\rho) < \frac{1}{t}\ln\left(\frac{\sigma(0)}{\sigma(1)}\right)$. Similar conditions can be obtained for A_2 also.

4.3 *System with constant mass and time dependent frequency*

Consider the q-deformed Hamiltonian

$$H_q = \frac{P_q^2}{2M} + \tfrac{1}{2}M\omega_0^2[1 + \chi\cos(2\omega_0 + \varepsilon)t]X_q^2. \tag{52}$$

The squeezing properties of the corresponding $q = 1$ Hamiltonians have been studied by Geethakumari *et al* [unpublished].

We put

$$\mu(t) = \tfrac{1}{2}\left[\left(\frac{1+\cos(2\omega_0+\varepsilon)t}{1+\chi}\right)^{1/2} + \left(\frac{1+\chi}{1+\cos(2\omega_0+\varepsilon)t}\right)^{1/2}\right],$$

$$\nu(t) = \tfrac{1}{2}\left[\left(\frac{1+\cos(2\omega_0+\varepsilon)t}{1+\chi}\right)^{1/2} - \left(\frac{1+\chi}{1+\cos(2\omega_0+\varepsilon)t}\right)^{1/2}\right]. \tag{53}$$

Then

$$(\Delta A_1)^2 = \tfrac{1}{4}\left[\frac{1+\chi}{1+\cos(\omega_0+\varepsilon)t}\right]\sigma(t),$$

$$(\Delta A_2)^2 = \tfrac{1}{4}\left[\frac{1+\chi}{1+\cos(2\omega_0+\varepsilon)t}\right]\sigma(t),$$

$$(\Delta A_1)(\Delta A_2) = \tfrac{1}{4}\sigma(t). \tag{54}$$

A_1 is squeezed if $\dfrac{1+\chi}{1+\cos(2\omega_0+\varepsilon)t} < \dfrac{1}{\sigma(t)}$ and

q-squeezed if $\dfrac{1+\chi}{1+\cos(2\omega_0+\varepsilon)t} < \dfrac{\sigma(0)}{\sigma(t)}$.

5. Results and discussion

We have calculated the probability distribution P_n for various values of q keeping α fixed. In figure 1, $\log_{10}P_n(q)$ is plotted against n for four different values of q, namely 1, 0·9, 0·8 and 0 for $\alpha = 0{\cdot}5$. Figure 2 gives the corresponding graph for $\alpha = 0{\cdot}9$. It is clear from these that the graphs for a given q value for different α values are similar.

In figure 3, $\log_{10}P_0(q)/P_0(1)$ is plotted against q where $P_0(q)$ is the probability of the ground state corresponding to $q \neq 1$ and P_0 that corresponding to $q = 1$. These graphs plotted for the values, namely, $\alpha = 0{\cdot}5$ and 0·9, show marked variation in their behaviour except near the standard value, namely, $q = 1$, where they converge.

Starting from a commutation relation for the q oscillator, coherent states of the oscillator have been constructed and the variances of the quadrature components calculated. The q analogues of bosonic squeezed states have been defined in two ways and are illustrated using three model Hamiltonians. It has been found that if we could achieve a condition which pushes q below unity, we could measure X_q or P_q (or both) with uncertainties smaller than $\hbar/2$. But ultimately, it is experiment that ought to determine the physical realizability of such states.

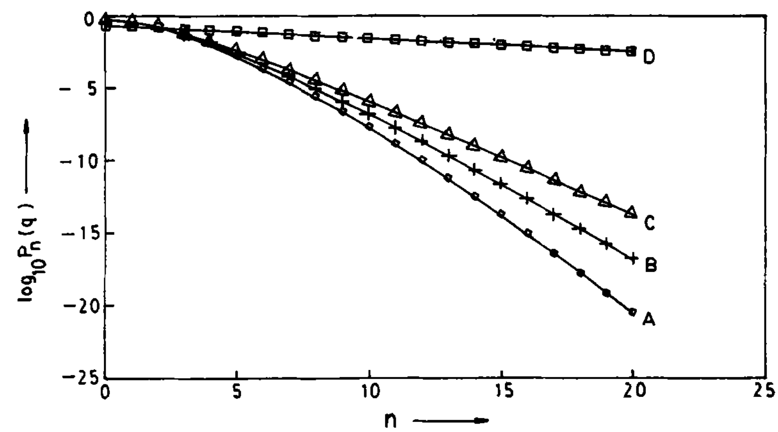

Figure 1. Variation of $\log_{10}P_n(q)$ with n, for $\alpha = 0{\cdot}5$: A) $q = 1$; B) $q = 0{\cdot}9$; C) $q = 0{\cdot}8$; D) $q = 0$.

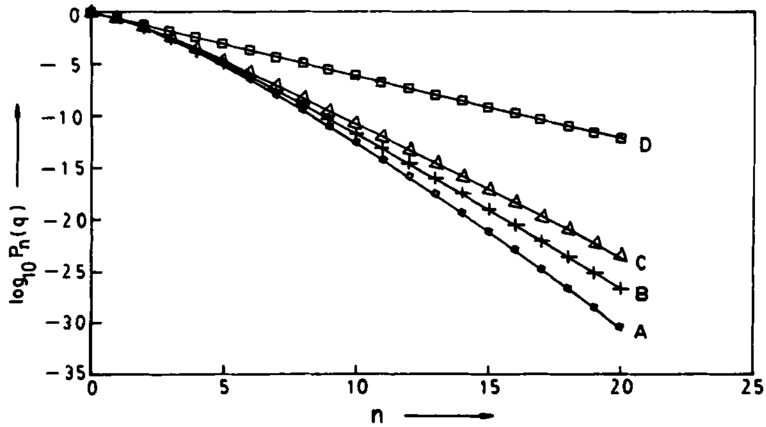

Figure 2. Variation of $\log_{10} P_n(q)$ with n, for $\alpha = 0.9$: A) $q = 1$; B) $q = 0.9$; C) $q = 0.8$; D) $q = 0$.

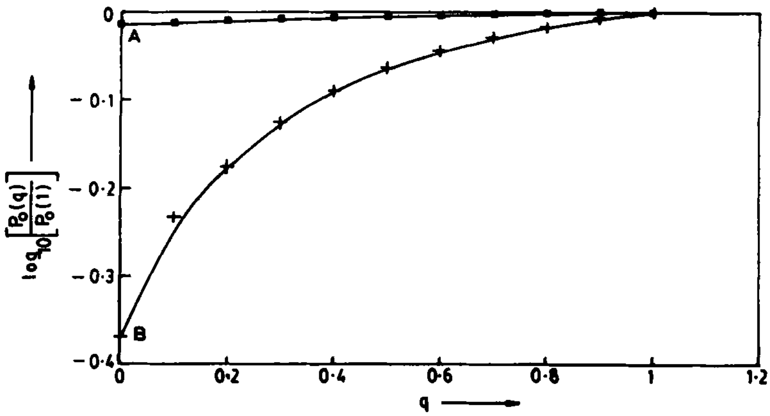

Figure 3. Variation of $\log_{10}[P_0(q)/P_0(1)]$ with q: A) $\alpha = 0.5$; B) $\alpha = 0.9$.

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