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Non perturbative effective potentials of quantum oscillators

ROSE P IGNATIUS and K BABU JOSEPH
Department of Physics, Cochin University of Science and Technology, Kochi 682022, India

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Abstract. Non perturbative analogues of the Gaussian effective potential (GEP) are defined for quantum oscillators obeying q—or (q,p)—deformed commutation relations. These are called the non perturbative q-effective potential (NP_qEP) and the non perturbative qp effective potential ($NP_{qp}EP$), in the respective cases. A system-specific effective potential (SSEP) is also introduced by means of an additional minimization with respect to the q or q and p parameters. The method is applied to q and (q,p) oscillators of the quartic and sextic types. The SSEP in the case of ground states of the q-oscillators corresponds to $q = 1$, which is the ordinary bosonic limit. A potential shape transition that involves the conversion of a double well to a single well or vice versa, is seen to exist in the case of quantum oscillators sitting in a double well potential.

Keywords. Quantum groups; q-oscillator; non-perturbative effective potential; anharmonic oscillators.

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1. Introduction

Quantum groups and quantum algebras have been receiving considerable attention in recent years [1–6]. Some of these investigations focus on quantum group modified quantum mechanics. In reference 6, for example, the spectrum of a q-anharmonic oscillator with quartic interaction has been studied using first order perturbation theory. There is a logical need to apply the non perturbative approach to such systems that are generically known as quantum oscillators.

It is clear by now that the Gaussian effective potential (GEP) provides a powerful method for accounting the effect of quantum fluctuations on the classical potential [7]. A non perturbative technique enables one to perform systematic approximations in quantum mechanics as well as quantum field theory [7–9]. In this work we formulate a non perturbative q—or (q,p)—analogue of GEP with the help of appropriate quantum oscillator commutation relations that depend on a single parameter q, or two parameters q,p. The resulting quantity is called the non perturbative q-effective potential (NP_qEP) or the non perturbative (q,p)—effective potential ($NP_{qp}EP$), as the case may be. In the original version of GEP there is a positive mass parameter Ω with respect to which the potential is minimized. When a quantum oscillator algebra is employed, the quantum parameters such as q,p can serve as additional parameters in the potential, suggesting a more elaborate scheme of minimization. The end product of the minimization procedure is termed the system-specific effective potential (SSEP). Following Barnes and Ghandour [10] the renormalized mass m_R^2 and coupling constant λ_R are calculated directly from the NP_qEP.

We study three kinds of quantum oscillator systems: quartic coupled quantum oscillators in a single well and in a double well, and sextic coupled quantum oscillators. Several interesting aspects emerge from these analyses. For example, for the ground state of a quartic or sextic anharmonic q-oscillator, the NP_qEP is a minimum corresponding to $q = 1$ and a maximum corresponding to $q = -1$. The renormalized mass m_R turns out to be a maximum at $q = 1$. Since m_R has the physical significance of being the first excitation energy, these observations seem to cast ordinary ($q = 1$) quantum mechanics in a new perspective. For the X^4-anharmonic (q, p)-oscillator, $NP_{qp}EP$ yields a minimum only if λ or $\hbar$ vanishes. In the case of quartic q or (q, p)-oscillator in a double well potential, critical values exist for q or q as well as p, for which the double well degenerates into a single well. Results for quantum oscillators with sextic interaction also exhibit some interesting features.

This paper is organized into five sections. In § 2 the concept of NP_qEP, $NP_{qp}EP$ and SSEP are introduced. Quartic quantum oscillators in single and double well are discussed in § 3. Sextic quantum oscillators constitute the theme of § 4 and concluding remarks comprise § 5.

2. Non perturbative effective potentials

It is well-known that quantum fluctuations modify the classical potential energy, the zero point energy being the prime example. To discuss these effects in the framework of q-deformed quantum mechanics, we seek generalizations of position and momentum operators, X and P, in the form

$$X = X_0 + \left(\frac{\hbar}{2\Omega}\right)^{1/2} (a_\xi + a_\xi^+) \tag{1}$$

$$P = \frac{-i}{2}(2\hbar\Omega)^{1/2}(a_\xi - a_\xi^+) \tag{2}$$

where X_0 is a classical c-number and a_ξ and a_ξ^+ are annihilation and creation operators, respectively, with the set $\xi = [\Omega, q]$. Here Ω denotes a variational parameter having the dimension of mass or frequency that appears in the original $q = 1$ formulation of the GEP [7], and q is a quantum deformation parameter. For $q = 1$ the standard definitions of X and P are recovered.

We impose the q-commutation relation:

$$a_\xi a_\xi^+ - q a_\xi^+ a_\xi = q^{-N_\xi} \tag{3}$$

where N_ξ is the number operator which is not assumed to be the same as $a_\xi^+ a_\xi$. In terms of X and P the q-commutation relation reads

$$[X, P] = i\hbar[q^{-N_\xi} + (q - 1)a_\xi^+ a_\xi]. \tag{4}$$

Although, in principle, q could be real or complex, consistency of (4) with the assumption that X and P are simultaneously hermitian, constrains q to be a real parameter. The number operator N_ξ is required to satisfy the commutation relations:

$$[a_\xi, N_\xi] = a_\xi, \tag{5}$$

$$[a_\xi^+, N_\xi] = -a_\xi^+. \tag{6}$$

The number eigenstates $|n\rangle_\xi$ are defined by the relation

$$N_\xi|n\rangle_\xi = n|n\rangle_\xi \tag{7}$$

where $n = 0, 1, 2\ldots$ The ground state $|0\rangle_\xi$ is assumed to be annihilated by a_ξ:

$$a_\xi|0\rangle_\xi = 0. \tag{8}$$

The nth excited state $|n\rangle_\xi$ is obtained from $|0\rangle_\xi$ by operating a_ξ^+, n times:

$$|n\rangle_\xi = \frac{(a_\xi^+)^n}{([n]!)^{1/2}}|0\rangle_\xi \tag{9}$$

where the q-factorial $[n]!$ is

$$[n]! = [n][n-1]\ldots\ [1]$$

with

$$[\mathrm{A}] = \frac{q^A - q^{-A}}{q - q^{-1}}.$$

The action of a_ξ and a_ξ^+ on the eigenkets $|n\rangle_\xi$ is postulated as follows:

$$a_\xi|n\rangle_\xi = [n]^{1/2}|n-1\rangle_\xi, \tag{10}$$

$$a_\xi^+|n\rangle_\xi = [n+1]^{1/2}|n+1\rangle_\xi. \tag{11}$$

The GEP is customarily defined [7] in the following fashion:

$$\begin{aligned} V_G(X_0) &= \min_\Omega V_G(X_0, \Omega), \\ &\equiv \min_\Omega \langle\psi|H|\psi\rangle, \end{aligned} \tag{12}$$

where $X_0 = \langle\psi|X|\psi\rangle$ and the lowest variational state $|\psi\rangle$ is a Gaussian,

$$|\psi\rangle = \left(\frac{\Omega}{\pi\hbar}\right)^{1/4} \exp\left[\frac{-\Omega}{2\hbar}(X - X_0)^2\right], \qquad \Omega > 0. \tag{13}$$

When q is real and $q > 1$, a q-analogue of the Gaussian function has been defined, [11] but its explicit form is not required here. We merely assume that the lowest variational trial state $|\psi\rangle_\xi$ depends on Ω as well as q, and the excited states can be generated therefrom by applying a_ξ^+ as many times as necessary. The $\mathrm{NP}_q\mathrm{EP}$, with respect to any state $|\psi\rangle_\xi$ is defined as follows:

$$\begin{aligned} V_q(X_0) &= \min_\Omega V_q(X_0, \Omega) \\ &\equiv \min_\Omega {}_\xi\langle\psi|H|\psi\rangle_\xi. \end{aligned} \tag{14}$$

Studies in quantum group phenomenology [12] indicate the possibility of a given system being associated with a particular q value. This motivates one to define the

SS_qEP as follows:

$$V(X_0) = \min_{\xi} V(X_0, \xi), \tag{15}$$

$$\equiv \min_{\Omega, q} {}_{\xi}\langle \psi | H | \psi \rangle_{\xi}.$$

If the two-parameter quantum algebra characterising a (q, p)-oscillator is used [13] then the relevant commutation relations are

$$a_\eta a_\eta^+ - q a_\eta^+ a_\eta = p^{-N_\eta}, \tag{16}$$

$$a_\eta a_\eta^+ - p^{-1} a_\eta^+ a_\eta = q^{N_\eta}, \tag{17}$$

where the set $\eta = [\Omega, q, p]$. A (q, p)-deformed number $[A]_{q,p}$ is defined by the relation

$$[A]_{q,p} = \frac{q^A - p^{-A}}{q - p^{-1}}.$$

It is clear that the q-oscillator corresponds to the particular case where $q = p$. By analogy with the NP_qEP, one defines the $NP_{qp}EP$ by the relation,

$$V_{q,p}(X_0) = \min_{\Omega} V_{q,p}(X_0, \Omega), \tag{18}$$

$$\equiv \min_{\Omega} {}_{\eta}\langle \psi | H | \psi \rangle_{\eta}.$$

The system-specific $V(X_0)$, denoted $SS_{qp}EP$, is defined as

$$V(X_0) = \min_{\eta} V(X_0, \eta), \tag{19}$$

$$= \min_{\Omega, q, p} {}_{\eta}\langle \psi | H | \psi \rangle_{\eta}.$$

The renormalized mass m_R and renormalized coupling constant λ_R (in the quartic case) are obtained by differentiating the effective potential:

$$m_R^2 = \left(\frac{d^2 V}{d X_0^2} \right)_{X_0 = 0}, \tag{20}$$

$$\lambda_R = \frac{1}{4!} \left(\frac{d^4 V}{d X_0^4} \right)_{X_0 = 0}. \tag{21}$$

It is clear that both m_R^2 and λ_R depend on the quantum parameter(s). m_R can be interpreted as the difference between the first excited state and ground state energies while λ_R denotes the amplitude for a transition from the state $|1\rangle$ to $|3\rangle$ under the action of coupling λX^4 [10].

3. Quartic quantum oscillators

3.1 *Single well*

The Hamiltonian representing a quartic quantum (single well) oscillator is

$$H = \tfrac{1}{2}P^2 + \tfrac{1}{2}m^2 X^2 + \lambda X^4. \tag{22}$$

If the system is a q-oscillator, then its $\mathrm{NP}_q\mathrm{EP}$ can be evaluated using the method developed in the preceding section. The expectation value of H for the nth eigenstate is written as

$$\langle n|H|n\rangle = \sum_{l=0,2,4} K_l X_0^l \tag{23}$$

where

$$\begin{aligned} K_0 &= \frac{\hbar\Omega}{4}([n]+[n+1]) \\ &\quad + \frac{\hbar m^2}{4\,\Omega}([n]+[n+1]) \\ &\quad + \lambda\left(\frac{\hbar}{2\Omega}\right)^2 ([n+1][n+2]+[n+1]^2+2[n][n+1] \\ &\qquad + [n]^2+[n][n-1]) \\ K_2 &= \tfrac{1}{2}m^2 + 3\lambda([n]+[n+1]) \\ K_4 &= \lambda. \end{aligned}$$

The condition for the potential to be a minimum with respect to Ω is the cubic equation

$$A\bar{\Omega}^3 + B\bar{\Omega} + C = 0, \tag{24}$$

where

$$\begin{aligned} A &= [n]+[n+1], \\ B &= -(m^2 + 12\lambda X_0^2), \\ C &= -2\hbar\lambda([n+1][n+2]+[n+1]^2 \\ &\qquad + 2[n+1][n]+[n]^2+[n][n-1]). \end{aligned}$$

Of the three roots of (24), the largest positive one, designated as $\bar{\Omega}$, is to be employed for setting up the effective potential. This procedure is an extrapolation from the usual $q = 1$ bosonic theory [7].

The $\mathrm{NP}_q\mathrm{EP}$ for the ground state is obtained as

$$\begin{aligned} V_q(X_0)_g &= \tfrac{1}{4}\hbar\bar{\Omega} + \tfrac{1}{2}m^2\left(X_0^2 + \frac{\hbar}{2}\bar{\Omega}\right) \\ &\quad + \lambda\left\{X_0^4 + \frac{6X_0^2\hbar}{2\bar{\Omega}} + \frac{\hbar^2}{(2\bar{\Omega})^2}([1]+[2])\right\} \end{aligned} \tag{25}$$

where $\bar{\Omega}$ is the largest root of the equation

$$\bar{\Omega}^3 - (m^2 + 12\lambda X_0^2)\bar{\Omega} - 2\hbar\lambda([1] + [2]) = 0. \tag{26}$$

Assuming $\bar{\Omega} \neq 0$, (25) may be rewritten in the form

$$V_q(X_0)_g = \frac{\hbar}{2}\bar{\Omega} + \tfrac{1}{2}m^2X_0^2 + \lambda X_0^4 - \frac{\hbar}{(2\bar{\Omega})^2}([1] + [2]). \tag{27}$$

The renormalized mass m_R which is equal to the first exitation energy $E_1 - E_0$, is obtained from $V_q(X_0)_g$:

$$m_R^2 = m^2 + \frac{6\hbar\lambda}{\bar{\Omega}_0}, \tag{28}$$

where

$$\bar{\Omega}_0 = \bar{\Omega}|_{X_0=0}.$$

The renormalized coupling constant is

$$\lambda_R = \left(1 - \frac{12\lambda\hbar}{3\bar{\Omega}_0^3 - m_R^2\bar{\Omega}_0}\right) \Bigg/ \left(1 + \frac{6\hbar\lambda}{3\bar{\Omega}_0^3 - m_R^2\bar{\Omega}_0}\right). \tag{29}$$

In order to evaluate the ground state SSEP, we extremise the ground state potential (25) with respect to q:

$$\frac{\hbar^2\lambda}{(2\bar{\Omega})^2}(1 - q^{-2}) = 0. \tag{30}$$

Since $\lambda \neq 0$, $\bar{\Omega} \neq \infty$, it follows that the extrema correspond to $q = \pm 1$. One readily checks that, for positive λ, the ground state potential is a minimum for $q = 1$ (giving $SS_qEP = GEP$) and a maximum for $q = -1$. However, the extremal condition for the potential for the nth excited state, when written out fully, is a complicated algebraic equation of the $2(2n + 1)$ degree in q. Some of its roots may represent minima, while others, maxima, corresponding to $q \neq \pm 1$.

We have studied the variation of m_R^2 given by (28) as a function of q (figure 1) and found that for $q > 0$, $m_R^2 > 0$ and that m_R^2 has a maximum at $q = 1$, the ordinary bosonic limit. If m^2 is not very much larger than λ, for all negative q values $\bar{\Omega}_0$ becomes negative. Recalling that in the ordinary bosonic theory the mass parameter Ω for the ground state GEP, is kept positive for convergence reasons [7], we are prompted to retain this proviso in the q-boson/(q,p)-boson theory, as well. Equally important is the positivity of m_R^2. However, for very small positive λ (compared to m^2), we obtain positive $\bar{\Omega}_0$ and positive m_R^2.

The effective potential for a quartic (single well) (q,p)-oscillator may be evaluated using the procedure sketched in the preceding section. Assuming the same Hamiltonian as given by (22), but invoking the (q,p)-commutation relations (16) and (17), one obtains the $NP_{qp}EP$ for any state $|n\rangle_{q,p}$. The potential is formally the same as NP_qEP with all the q-deformed brackets [ ] replaced by the corresponding (q,p)-deformed ones $[\ ]_{q,p}$. The $NP_{qp}EP$ is finally derived by minimization with respect to Ω.

In the case of q-oscillators, it has been mentioned above that the SS_qEP for the ground state is same as the GEP. For a (q,p) oscillator the ground state effective

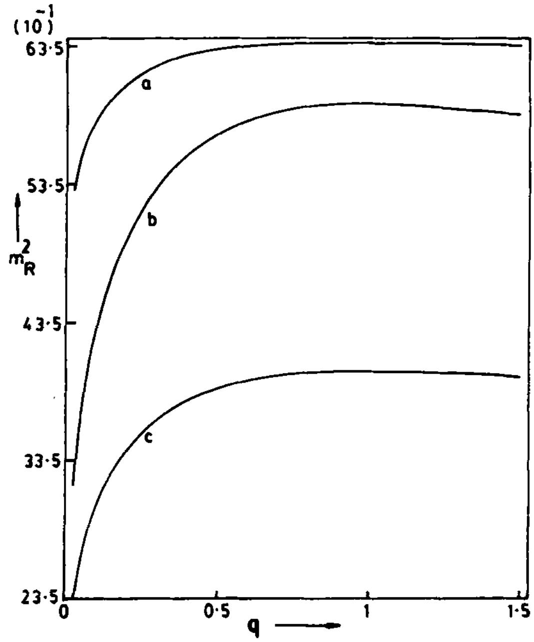

Figure 1. Variation of the renormalized mass m_R^2 with the q parameter for a q-oscillator moving in a quartic potential well.
a) $m = 2$ $\lambda = 1$; b) $m = 1$ $\lambda = 2$; c) $m = 1$ $\lambda = 1$.

potential is a minimum only if

$$\frac{\lambda\hbar^2}{(2\bar{\Omega})^2} = 0 \tag{31}$$

and

$$\frac{\lambda\hbar^2}{(2\Omega p)^2} = 0. \tag{32}$$

These relations imply that either $\lambda = 0$ or $\hbar = 0$. Since the latter condition can be easily ruled out, one is confronted with the possibility of a trivial, non-interacting (q, p)-oscillator theory. The message is clear: the ground state $SS_{qp}EP$ for a quartic (q, p)-oscillator cannot be found by the variational method herein presented. These remarks, however, need not apply to the excited states.

3.2 *Double well*

The Hamiltonian for a double well quartic potential is chosen in the form

$$H = \tfrac{1}{2}P^2 + \tfrac{1}{2}m^2X^2 + \lambda X^4 + \frac{m^4}{16\lambda} \tag{33}$$

where $m^2 < 0$, $\lambda > 0$. As this differs from the single well Hamiltonian only by the presence of the constant term, the evaluation of the nonperturbative effective potential is not a novel exercise.

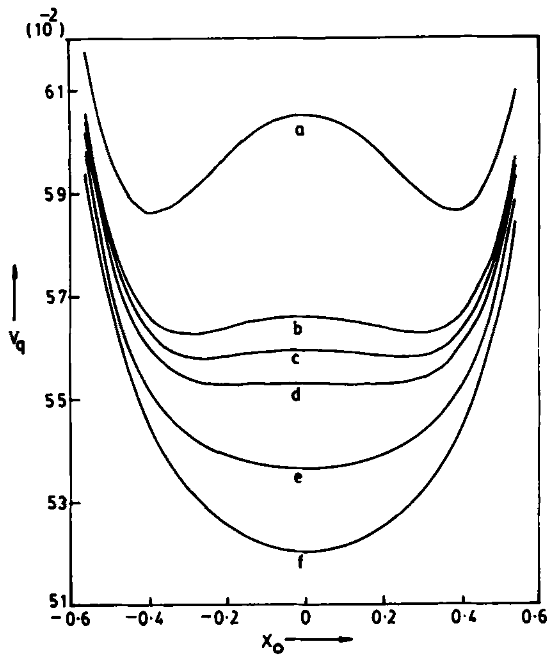

Figure 2. Variation of V_q of a q-oscillator moving in a double well potential, with position X.
a) $q = 0{\cdot}2$; b) $q = 4$; c) $q = 0{\cdot}26$; d) $q = 3{\cdot}7$; e) $q = 0{\cdot}3$; f) $q = 3$.

The only interesting point that crops up is the existence of critical parameter values that separate the double well region from the single well region. This becomes evident from a numerical study. Fixing $\lambda = 1$ and $(m^2/\lambda) = -2$ it is seen that, for a q oscillator, there is a critical value for q, denoted by $q_{\text{crit }1}$, above which the double well shape degenerates into a single well shape and another critical value $q_{\text{crit }2}$ at which the double well shape is regained. In the present case, $q_{\text{crit }1} \approx 0{\cdot}16$ and $q_{\text{crit }2} \approx 6{\cdot}27$. Again with $\lambda = 2$, $q_{\text{crit }1} \approx 0{\cdot}26$ and the second critical value $q_{\text{crit }2} \approx 3{\cdot}66$ at which the double well shape is recovered. The variation of V_q vs X for various values of q parameter, showing the details of the potential shape transitions, is sketched in figure 2.

Critical behaviour is exhibited also by (q, p)-oscillators moving in a double well potential. In this case, one varies both q and p. For instance, taking $\lambda = 2$, $m^2/\lambda = -2$, $q = 0{\cdot}2$, we have obtained double well behaviour in the domain $-0{\cdot}83 \lesssim p \lesssim 0{\cdot}26$. However, for $-0{\cdot}83 \lesssim p < 0$, $\bar{\Omega}_0$ is negative and for $0 < p \lesssim 0{\cdot}25$. $\bar{\Omega}_0$ is positive. In the single well region corresponding to $p \lesssim -0{\cdot}83$, $\bar{\Omega}$ is positive. Similar critical behaviour may be monitored, alternatively, by keeping p fixed and tuning q.

4. Sextic quantum oscillators

A general sextic anharmonic oscillator is modelled by the Hamiltonian

$$H = \tfrac{1}{2}P^2 + \sum_{j=1}^{6} C_j X^j. \tag{34}$$

Assuming q-commutation relations, the expectation value of the Hamiltonian for the

nth state is obtained:

$$\langle n|H|n\rangle = \sum_{l=0}^{6} C_l g_l \tag{35}$$

where C_0 is a constant that depends only on Ω and q parameters:

$$C_0 = \frac{\hbar\Omega}{4}([n] + [n+1]).$$

The remaining coefficients $C_1 (l \neq 0)$ have the same significance as in (34). The functions g_1 are given by the following set of relations:

$$g_0 = 1$$

$$g_1 = X_0$$

$$g_2 = X_0^2 + \frac{\hbar}{2\Omega}([n] + [n+1])$$

$$g_3 = X_0^3 + \frac{3\hbar}{2\Omega} X_0([n] + [n+1])$$

$$g_4 = X_0^4 + \frac{\hbar}{2\Omega} X_0^2 6([n] + [n+1]) + \left(\frac{\hbar}{2\Omega}\right)^2 ([n+2][n+1] + [n+1]^2 + 2[n+1][n] + [n]^2 + [n][n-1])$$

$$g_5 = X_0^5 + \frac{10X_0^3\hbar}{2\Omega}([n] + [n+1]) + \frac{5X_0\hbar^2}{(2\Omega)^2}([n+2][n+1] + [n+1]^2 + 2[n+1][n] + [n]^2 + [n][n-1])$$

$$g_6 = X_0^6 + \frac{15X_0^4\hbar}{2\Omega}([n] + [n+1]) + 15X_0^2\left(\frac{\hbar}{2\Omega}\right)^2 ([n+2][n+1] + [n+1]^2 + 2[n+1][n] + [n]^2 + [n][n-1]) + \left(\frac{\hbar}{2\Omega}\right)^3 ([n+3][n+2] + [n+1] + [n+2]^2[n+1] + 2[n+2][n+1]^2 + 2[n+2][n+1][n] + [n+1]^3 + 2[n+1][n-1][n] + 3[n+1]^2[n] + 3[n+1][n]^2 + [n]^3 + 2[n]^2[n-1] + [n][n-1]^2 + [n][n-1][n-2]).$$

The condition for the optimum $\Omega(\bar{\Omega})$ is expressed as

$$D\bar{\Omega}^2 + \hbar E\bar{\Omega} + \hbar^2 F = 0, \tag{36}$$

where the coefficients are

$$D = -(2C_2 + 6C_3X_0 + 12C_4X_0^2 + 20C_5X_0^3 + 30C_6X_0^4)$$

$$E = -(C_4 + 5C_5X_0 + 15C_6X_0^2)([n+2][n+1] + [n+1]^2 + 2[n+1][n] + [n]^2 + [n][n-1])/([n] + [n+1])$$

$$F = \frac{-3}{2} C_6([n+3][n+2][n+1] + [n+2]^2[n+1] + 2[n+2][n+1]^2 + 2[n+2][n+1][n] + [n+1]^3 + 2[n+1][n-1][n] + 3[n+1]^2[n] + 3[n+1][n]^2 + [n]^3 + 2[n]^2[n-1] + [n][n-1]^2 + [n][n-1][n-2]).$$

The ground state expectation value of the Hamiltonian is

$$\langle 0|H|0\rangle = f_0 + \sum_{k=1}^{6} C_k f_k \tag{37}$$

where

$$f_0 = \frac{\hbar\Omega}{4} + \left(\frac{\hbar}{2\Omega}\right)^3 C_6([1] + 2[2] + [2]^2 + [2][3])$$

$$f_1 = X_0,$$

$$f_2 = X_0^2 + \frac{\hbar}{2\Omega},$$

$$f_3 = X_0^3 + \frac{3\hbar}{2\Omega} X_0,$$

$$f_4 = X_0^4 + \frac{6\hbar}{2\Omega} X_0^2 + \left(\frac{\hbar}{2\Omega}\right)^2 ([1] + [2]),$$

$$f_5 = X_0^5 + \frac{10\hbar}{2\Omega} X_0^3 + 5X_0 \left(\frac{\hbar}{2\Omega}\right)^2 ([2] + [1]),$$

$$f_6 = X_0^6 + \frac{15}{2\Omega} \hbar X_0^4 + 15\left(\frac{\hbar}{2\Omega}\right)^2 ([1] + [2]) X_0^2.$$

The optimization condition is

$$\bar{\Omega}^4 + G\bar{\Omega}^2 + X\bar{\Omega} + I = 0, \tag{38}$$

with the symbols standing for the following:

$$G = -(2C_2 + 6C_3X_0 + 12C_4X_0^2 + 20C_5X_0^3 + 30C_6X_0^4)$$

$$H = -\hbar(2C_4 + 10C_5X_0 + 30C_6X_0^2)([1] + [2]).$$

$$I = \frac{-3}{2} C_6 \hbar^2([1] + 2[2] + [2]^2 + [2][3]).$$

Denoting the largest positive root of (38) at $X_0 = 0$ by $\bar{\Omega}_0$, one writes an expression for the renormalized mass:

$$m_R^2 = 2C_2 + 12C_4\left(\frac{\hbar}{2\bar{\Omega}_0}\right) + 30C_6([1] + [2])\left(\frac{\hbar}{2\bar{\Omega}_0}\right)^2 - \left\{3C_3\frac{\hbar}{2\bar{\Omega}_0} + 5C_5\frac{\hbar^2}{2\bar{\Omega}_0^2}([2] + [1])\right\}\frac{d\bar{\Omega}}{dX_0}\bigg|_{X_0=0} \tag{39}$$

where

$$\left.\frac{d\bar{\Omega}}{dX_0}\right|_{X_0=0} = \frac{6C_3\bar{\Omega}_0 + 5C_5\hbar([2]+[1])}{4\bar{\Omega}_0^3 - 4C_2\bar{\Omega}_0 - C_4\hbar([2]+[1])}.$$

Numerical computations show that when q is negative, for positive coefficients $C_1, \ldots C_6$, and C_6 not very much smaller than C_2, all the roots of the quartic equation (38) are imaginary. Under the same conditions, and for positive q values, two real roots exists, of which one is positive and the other negative.

Setting the odd order coefficients equal to zero, we obtain a sextic oscillator with only even powers of X in the potential. In this case the renormalized mass is

$$m_R^2 = 2C_2 + \frac{6\hbar C_4}{\bar{\Omega}_0} + 30C_6\left(\frac{\hbar}{2\bar{\Omega}_0}\right)^2([1]+[2]). \tag{40}$$

A plot of m_R^2 vs positive q values, is quite similar to that for the X^4 theory. Taking only even order coefficients as positive, and C_6 not very much smaller than C_2, for negative q values the theory is not defined, because all the roots of the $\bar{\Omega}$ equation (38) become imaginary. This shows that the quantum sextic model possesses physical significance only in selected parameter domains.

One can define renormalized quartic (C_{4R}) and renormalized sextic (C_{6R}) coupling constants in respect of the even power sextic q-oscillator model. Thus

$$\begin{aligned}C_{4R} &= \frac{1}{4!}\left.\frac{d^4 V_q}{dX_0^4}\right|_{X_0=0}\\ &= C_4 - \frac{3C_4}{24}\frac{\hbar}{\bar{\Omega}_0^2}\left.\frac{d^2\bar{\Omega}}{dX_0^2}\right|_{X_0=0} + 15C_6\frac{\hbar}{2\bar{\Omega}_0}\\ &\quad - \frac{15}{8}C_6\frac{\hbar^2}{\bar{\Omega}_0^3}([1]+[2])\left.\frac{d^2\bar{\Omega}}{dX_0^2}\right|_{X_0=0}\end{aligned} \tag{41}$$

and

$$\left.\frac{d^2\bar{\Omega}}{dX_0^2}\right|_{X_0=0} = \frac{24\bar{\Omega}_0^2C_4 + 60C_6\hbar\bar{\Omega}_0([1]+[2])}{4\bar{\Omega}_0^3 - 4C_2\bar{\Omega}_0 - 2C_4\hbar([1]+[2])}.$$

Expressing the fourth derivative of $\bar{\Omega}$ at $X_0 = 0$, in the form

$$\begin{aligned}\left.\frac{d^4\bar{\Omega}}{dX_0^4}\right|_{X_0=0} &= \left\{-720C_6\bar{\Omega}_0^2 - 2[144C_4\bar{\Omega}_0 + 180C_6([1]+[2])\left.\frac{d^2\bar{\Omega}}{dX_0^2}\right|_{X_0=0}\right.\\ &\quad \left. + (36\bar{\Omega}_0^2 - 12C_2)\left(\frac{d^2\bar{\Omega}}{dX_0^2}\right)_{X_0=0}\right\}\Big/(4\bar{\Omega}_0^3 - 4\bar{\Omega}_0C_2 - 2C_4\hbar([1]+[2]))\end{aligned}$$

$$\begin{aligned}C_{6R} &= \frac{1}{6!}\left.\frac{d^6 V_q}{dX_0^6}\right|_{X_0=0}\\ &= C_6\frac{-5C_6\hbar}{2\bar{\Omega}_0^2} - \frac{\hbar}{2\bar{\Omega}_0^2}\left[\frac{1}{12}C_4 + \frac{\hbar}{2\bar{\Omega}_0}([1]+[2])\frac{10C_6}{24}\right]\left(\frac{d^4\bar{\Omega}}{dX_0^4}\right)_{X_0=0}\\ &\quad + \frac{\hbar}{2\bar{\Omega}_0^3}\left[\frac{C_4}{2} + \frac{\hbar}{2\bar{\Omega}_0}\frac{15C_6}{4}([1]+[2])\right]\left(\frac{d^2\bar{\Omega}}{dX_0^2}\right)_{X_0=0}\end{aligned} \tag{42}$$

To get the system-specific q effective potential (SS_qEP), one has to determine the optimization condition for q also.

Differentiating $\langle 0|H|0\rangle$ given by (37) with respect to q, we have

$$(1-q^{-2})\left(\frac{\hbar}{2\bar{\Omega}}\right)^2\left\{C_4+5C_5X_0+C_6\left(15X_0^2+\frac{\hbar}{2\bar{\Omega}}(q^{-2}-q^{-1}+1)(1+q^2+q^{-2}+2)\right)\right\}=0. \quad (43)$$

This equation has the roots $q=\pm 1$.
But

$$\frac{d^2V_q}{dq^2}=(1+2q^{-3})\left(\frac{\hbar}{2\bar{\Omega}}\right)^2\left\{C_4+5C_5X_0+C_6\left(15X_0^2+\frac{\hbar}{2\bar{\Omega}}(q^{-4}-q^{-3}+2q^{-2}-q^{-1}+q^2-q+2)\right)\right\}.$$

For posivive coefficients, this equation becomes positive at $q=1$, showing that it is a minimum of the potential. If $q=-1$ it becomes negative, and hence corresponds to the maximum of the effective potential. q can have another set of six values corresponding to the roots of the factored out expression, which depend on the coefficients C_k. The possibility of some of them representing true minima, cannot be ruled out.

As for the (q,p) analogue of the sextic oscillator, we have equations (35), (36), (37) and (38) with $[A]$ replaced by $[A]_{q,p}$. In order to get the $SS_{qp}EP$ for the ground state, the conditions are

$$\frac{\partial V_{q,p}}{\partial p}=\left(\frac{\hbar}{2\bar{\Omega}}\right)^2(-p^{-2})\left[C_4+5C_5X_0+15C_6X_0^2+\frac{C_6\hbar}{2\bar{\Omega}}(3p^{-2}+2q^2+4qp^{-1}+2q+2p^{-1}+2)\right]=0 \quad (44)$$

$$\frac{\partial V_{q,p}}{\partial q}=\left(\frac{\hbar}{2\bar{\Omega}}\right)^2\left[C_4+5C_5X_0+15X_0^2C_6+C_6\frac{\hbar}{2\bar{\Omega}}(3q^2+2p^{-2}+4qp^{-1}+2q+2p^{-1}+2)\right]=0 \quad (45)$$

besides equation (38).

If $p=\infty$ then the conditions (44) and (45) imply $q=0$. Non-trivial solutions of (44) and (45) correspond to $p^{-2}\neq 0$. The $SS_{qp}EP$ corresponds to $q=p^{-1}$ subject to the condition that the second derivatives of $V_{q,p}$ are greater than or equal to zero.

5. Concluding remarks

In this paper we have addressed the question of formulating q and (q,p) analogues of the GEP. A direct generalization of GEP gives the non perturbative effective potentials, namely NP_qEP and $NP_{qp}EP$, applicable to q-oscillators and (q,p)-oscillators, respectively. The SS_qEP is seen to correspond to $q = 1$, at least in the ground state of q oscillators, showing that the ordinary bosonic theory appears to have a natural significance in the variational approach. Such uniqueness is not necessarily shared by the excited states of the system.

The potential shape transitions exhibited by double well oscillators at critical values of the parameter(s), is a novel phenomenon which may have implications in the study of spontaneous symmetry breaking in q and (q, p)-quantum field theories.

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