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The Maxwell tensor for the Liénard–Wiechert field as an exact divergence

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Abstract. A potential for the bounded part of the Maxwell tensor associated to the Liénard-Wiechert field has been obtained by van Weert. In this paper we explicitly derive the non-local potential for the radiative part of such a tensor.

Keywords. Liénard-Wiechert field; superpotential for the Maxwell tensor; non-local potential for the electromagnetic field.

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1. Introduction

In what follows we will consider the Maxwell field generated by a point charge in Minkowski space. In order to include radiation reaction effects [1–3] on the charge motion it is necessary to compute the electromagnetic energy and momentum fluxes through hypersurfaces (for instance, the Bhabha tube [4, 5]), and it is in these computations where the great usefulness of the superpotential K_{abc} for the Maxwell tensor T_{ab} associated to the Liénard–Wiechert (LW) field [2] is evident.

The concept of superpotential is due to von Freud [6, 7] since he obtained a superpotential for the Einstein energy-momentum canonical pseudotensor [7], greatly used to study the gravitational energy in curved spaces, keeping the idea that if a physical quantity has a vanishing divergence there must exist a potential that generates that quantity (see reference [8]).

In §2 we present a brief summary of the LW field properties, emphasizing the corresponding tensor T_{ab} which admits the splitting [9]

$$T_{ab} = T_{Bab} + T_{Rab} \tag{1a}$$

where T_{Bab} and T_{Rab} are the bounded and radiative parts. These are dynamically independent because they separately satisfy the continuity equation [9] (sum is implied by repeated indices)

$$T^{b}_{Ba},_{b} = 0 \tag{1b}$$

$$T^{b}_{Ra},_{b} = 0 \tag{1c}$$

and, so, it is evident that $T^{b}_{a},_{b} = 0$.

Then, following von Freud [6] it is natural to look for a superpotential $K_{Babc} = -K_{Bbac}$ that generates the bounded part

$$T_{Bab} = K^{j}_{Bab,j} \quad \text{with} \quad j = \frac{\partial}{\partial x^j} \tag{2a}$$

from where (1a) follows immediately. Indeed, this was done by van Weert [10] who also showed that (2a) simplifies the computation of the T_{Bab} fluxes through a 3-surface around the charge. Van Weert did not discover the physical meaning of his superpotential, but it was done later [11] where it was found that K_{Babc} is the intrinsic angular momentum density [12] of the electromagnetic field produced by the point charge.

The next step is to construct a superpotential $K_{Rabc} = -K_{Rbac}$ for the radiative part

$$T_{Rab} = K^{j}_{Rab,j} \tag{2b}$$

and this will be done, using Minkowskian coordinates, in § 3. The K_{Rabc} thus obtained results to be non-local since it contains integrals over the past history of the charge, consistently with Rohrlich [13] statement: "... the process of measuring the radiation rate is intrinsically non-local".

Therefore, (1a), (2a) and (2b) imply

$$T_{bc} = (K^{j}_{Bbc} + K^{j}_{Rbc})_{,j} \tag{2c}$$

and the Maxwell tensor is an exact divergence.

2. The Liénard–Wiechert field

In special relativity a "particle" means a time-like world-line [14] (see figure 1) whose tangent vector is the 4-velocity

$$v^r = \frac{dq^r}{d\tau}, \quad v^r v_r = -1 \tag{3}$$

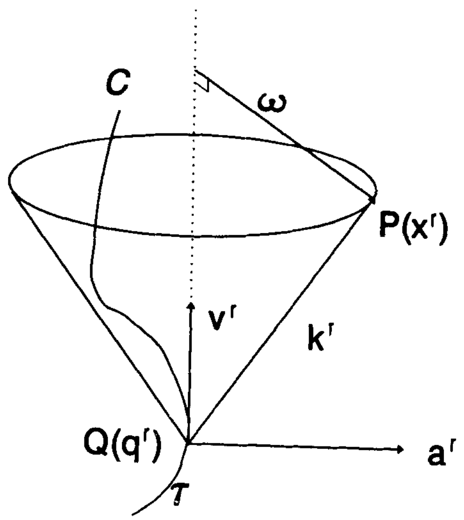

Figure 1. Kinematics of the world line.

where the proper time τ is defined by

$$d\tau^2 = -g_{ab}dx^a dx^b = -dx^2 - dy^2 - dz^2 + dt^2 \tag{4}$$

i.e., the metric is Diag$(1, 1, 1, -1)$ and c (speed of light in vacuum) $= 1$. Then, the 4-acceleration can be written as:

$$a^r = \frac{dv^r}{d\tau}, \quad a^r v_r = 0 \tag{5a}$$

which implies that a^r is space-like:

$$a^2 \equiv a^r a_r \geqslant 0. \tag{5b}$$

If from an event P with coordinates $(x^r) = (x, y, z, t)$ not on the path C we draw the past sheet of its null cone it will intersect C in the point $Q(q^r)$ called the "retarded event associated to P" and

$$q^r = q^r(x^b) \tag{6}$$

because if P is given the retarded point on C is automatically determined as the intersection of C and the null cone with vertex at P. The null vector

$$K^c = x^c - q^c, \quad K^c K_c = 0. \tag{7}$$

For any P we may construct (see figure 1) the retarded distance ω from P to C as proposed by Bhabha [4, 5, 11] and whose expression is:

$$\omega = -K^c v_c \geqslant 0 \tag{8a}$$

Recalling [14] that a null vector cannot be orthogonal to a time-like vector, we obtain from (8a) that:

$$\omega = 0 \text{ if and only if } K^r = 0. \tag{8b}$$

In performing the calculations we will need to know how the physical quantities on C change, but it suffices to have the law of change for τ, since q^r, v^r, a^r, etc depend on that parameter:

$$\tau_{,j} = -\omega^{-1} K_j. \tag{9}$$

All the events P on a given cone are associated with a unique value of τ. This is why the light-cone is the surface τ = constant and, hence, $\tau_{,j}$ is the vector normal to the cone.

In view of eq. (9) it is simple to obtain the useful relations:

$$q^r{}_{,j} = -\omega^{-1} v^r K_j \qquad v^r{}_{,j} = -\omega^{-1} a^r K_j$$
$$K^r{}_{,j} = \delta^r_j + \omega^{-1} v^r K_j, \quad \omega_{,j} = -v_j + BK_j \tag{10a}$$

where Plebañski's B invariant [15]:

$$B = \omega^{-1}(1 - W) \text{ with } W = -K^r a_r \tag{10b}$$

appears. Then

$$B_{,c} = \omega^{-1}[U_c - (B^2 + \omega^{-1} K^r S_r) K_c] \tag{10c}$$

where $S^r \equiv \mathrm{d}a^r/\mathrm{d}\tau$ is the superacceleration [5] and U_c is Synge's vector [5]

$$U_c \equiv Bv_c + a_c. \tag{10d}$$

The retarded 4-potential A^r generated by the point charge is the LW field [2, 14, 15] given by:

$$(A^r) = (\mathbf{A}, \phi) = (q\omega^{-1}v^r), \quad q = \frac{\text{charge}}{4\pi\varepsilon_0} \tag{11}$$

which generates the electric and magnetic fields through the well-known relations

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{12a}$$

Recalling the matrix representation of the Faraday tensor $F_{ab} = -F_{ba}$ [2, 7]

$$[F_{ab}] = \begin{bmatrix} 0 & B_z & -B_y & -E_x \\ -B_z & 0 & B_x & -E_y \\ B_y & -B_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{bmatrix} \tag{12b}$$

equations (12a) are equivalent to the tensor expression

$$F_{rc} = A_{c,r} - A_{r,c} \tag{13a}$$

and the Maxwell equations become

$$F^{rc}_{\ ,c} = 0, \quad F_{br,c} + F_{rc,b} + F_{cb,r} = 0 \tag{13b}$$

or, alternatively.

$$\nabla\cdot\mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{13c}$$

Substitution of (10a) and (11) into (13a) yields the Faraday tensor for the LW field [5]

$$F_{bc} = q\omega^{-2}(U_b K_c - U_c K_b) = q\omega^{-2} U_b \times U_c \tag{14a}$$

where we have used Lowry's notation [16]

$$A_b \times B_j \equiv A_b B_j - A_j B_b \tag{14b}$$

With the help of (9), (10c) and (14c) it is easy to prove that

$$F_{rb} = q\tau_{,r} \times B_{,b} \tag{14c}$$

which was first obtained by Plebañski [15].

The null vector K^r given by (7) is an eigenvector of (14a) and (14c), i.e.

$$F_{rb}K^b = q\omega^{-2}K_r. \tag{15}$$

Another vector important in electrodynamics of charged particles is [5]

$$p^c \equiv \omega^{-1} K^c - v^c \tag{16a}$$

which, by virtue of (3), (7) and (8a) satisfies

$$p^c p_c = 1,\ p^c v_c = 0 \tag{16b}$$

implying that p^b is unitary and space-like. Furthermore, with (10a) and (16a) it is simple to obtain that

$$p^r_{,j} = \omega^{-1}\delta^r_j + \omega^{-2} v_j K^r + \omega^{-1}(a^r + \omega^{-1} v^r - \omega^{-1} B K^r) K_j \tag{16c}$$

that, together with (14a), implies

$$F^j_c p^r_{,j} = q\omega^{-4}\, W v^r K_c \tag{16d}$$

Expressions (15) and (16a) are very valuable in deriving the superpotential K_{Rabc} for the radiative part of T_{ab}.

It is useful to have an orthonormal tetrad $e_{(a)c}$, $a = 1, \ldots, 4$ on C, because such a tetrad allows the construction of a basis for any tensor object on C. Here, we will use $e^r_{(4)} = v^r$ and the unitary space-like triad $e_{(\gamma)c}$, $\gamma = 1, 2, 3$ will be a "Fermi triad" [17, 18], that is, that it evolves on C according to the transport law:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} e^c_{(\gamma)} = a_{(\gamma)} v^c, \quad a_{(\gamma)} = a^b e_{(\gamma)b} \tag{17}$$

In this equation $a_{(\sigma)}$ are the projections of the acceleration a^r over the spacial triad $e^r_{(\sigma)}$, recalling that (5a) implied the space-like character of a^r. Similarly, we may project (16a) over this triad

$$p_{(\sigma)} = p^r e_{(\sigma)r} \tag{18a}$$

so that (16d), (17) and (18a) imply the equation

$$F^j_b p_{(\gamma),j} = 0 \tag{18b}$$

which will be of fundamental importance in § 3.

All the information about the electromagnetic field energy and momentum is contained in the Maxwell tensor defined [2, 7, 14] by

$$T_{ab} = T_{ba} \equiv F_{ac} F^c_b - (F_{ij} F^{ij}) g_{ab} \tag{19}$$

and substitution of (14a) into (19) leads to Teitelboim splitting [9] indicated in our equation [1a] such that

$$T_{Bbc} = e^2\omega^{-4}\left[\frac{1}{2} g_{bc} + (K_b a_c + K_c a_b) + \tilde{B}(K_b v_c + K_c v_b) - \omega^{-2}(1 - 2W) K_b K_c\right] \tag{20a}$$

and

$$T_{Rbc} = e^2\omega^{-4}(a^2 - \omega^{-2} W^2) K_b K_c \tag{20b}$$

satisfying (1b) and (1c) away from the world-line.

In reference [10] it is not mentioned how eq. (2a) was solved, but in [19, 20] the Newman–Penrose formalism [21] was used to establish the process by which K_{Babc}

was obtained, and it was found that

$$K_{B^b{}_{jc}}=\frac{q^2}{4}\omega^{-4}[\omega^{-1}(4W-3)(v_b\times K_j)K_c-4(a_b\times K_j)K_c-g_{ci}K_b+g_{cj}K_b] \tag{21a}$$

leads to (20a) through (2a). In the introduction we have already mentioned that the physical meaning of (21a) was determined in [11]. $K_{B_{abc}}$ is said to be local because it depends only on quantities associated to the two events P and Q shown in figure 1. In §3 we will see that $K_{R_{abc}}$ will not possess such a locality because it depends on integrals along the world-line of the charge.

Finally, we will mention an interesting identity, related to (21a) and first found by Rowe [22]

$$K_{B^i{}_{br}}=\left[\frac{q^2}{4}\omega^{-4}(g_{ri}K_bK^c+\delta^c_bK_rK_i-g_{br}K_iK^c-\delta^c_iK_bK_r)\right]_{,c} \tag{21b}$$

3. The potential for the radiative part

The main goal of our work consists of the construction of a superpotential $K_{R_{abc}}$ satisfying (2b). In this respect, the splitting of $T_{R_{ab}}$ introduced in [23] where it was proved that (20b) can be written as

$$T_{R_{bc}}=\bar{T}_{R_{bc}}+\tilde{T}_{R_{bc}} \tag{22a}$$

where

$$\bar{T}_{R_{bc}}=e^2\omega^{-4}(a^2-3\omega^{-2}W^2)K_bK_c \tag{22b}$$

$$\tilde{T}_{R_{bc}}=2e^2\omega^{-6}W^2K_bK_c \tag{22c}$$

satisfying the conservation laws

$$\bar{T}^b_{R^c{}_{,b}}=0,\quad \tilde{T}^b_{R^c{}_{,b}}=0 \tag{22d}$$

The physical meaning of the splitting (22a) is the following: As we enclose C by a Bhabha tube [4, 5] and calculate the electromagnetic energy and momentum fluxes of $\bar{T}_{R_{ab}}$ across this tube, it is found that the fluxes vanish. This means that if the Bhabha surface is used to determine the equation of motion [5] the tensor (22b) will not contribute with any term to such equation. Hence, the fluxes of (22a) across the Bhabha cylinder are due only to (22c).

Therefore, to solve (2b) is equivalent to finding solutions to

$$\bar{T}_{R_{bc}}=\bar{K}^j_{R_{bc,j}}, \tag{23a}$$

$$\tilde{T}_{R_{bc}}=\tilde{K}^j_{R_{bc,j}}, \tag{23b}$$

from which (22c) follows immediately. We can apply the method presented in references [19, 20] to (23a) to obtain

$$\bar{K}_{R^b{}_{jc}}=\frac{q^2}{4}\omega^{-2}[\omega^{-2}W^2(g_{cj}K_b-g_{cb}K_j)+\omega^{-1}(v_b\times K_j)(3\omega^{-2}WK_c-a_c)$$
$$+(a_b\times K_j)(a_c-4\omega^{-2}WK_c)] \tag{24a}$$

which is local in the same sense as $K_{B_{abc}}$.

On the other hand, there is no systematic and direct method to solve (23b). We were guided by the analysis of (9), (13b), (15), (17), (18a) and (18b) to construct the non-local superpotential

$$\tilde{K}_{R^{bjc}} = -2eF_{bj}p_{(\sigma)}p_{(\gamma)}\left[\int_0^\tau a_{(\sigma)}a_{(\gamma)}v_c\,\mathrm{d}\tau + p_{(\beta)}\int_0^\tau a_{(\sigma)}a_{(\gamma)}e_{(\beta)c}\,\mathrm{d}\tau\right] \tag{24b}$$

with sums over $\sigma, \beta, \gamma = 1, 2, 3$ are implied. The presence of integrals over the past history of the charge must be noticed. Summarizing, (2b) is solved by

$$K_{R^{bjc}} = \bar{K}_{R^{bjc}} + \tilde{K}_{R^{bjc}} \tag{24c}$$

and (2c) is proved.

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