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An identity for R_n embedded into E_{n+1}

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Abstract. An identity is obtained for a Riemannian n-space (R_n) locally and isometrically embedded into a pseudo-Euclidean $(n+1)$-space (E_{n+1}), relating the corresponding second fundamental form with the intrinsic geometry of R_n. For $n = 4$ such an identity reduces to a previous result by Goenner.

Keywords. Riemannian spaces embedding; local and isometric embedding.

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1. Introduction

We will say that R_n is locally and isometrically embedded into E_{n+1} if and only if there exists the second fundamental form tensor $b_{ac} = b_{ca}$ that satisfies the equations [1]

$$R_{ijkc} = \varepsilon(b_{ik}b_{jc} - b_{ic}b_{jk}) \qquad \text{Gauss} \tag{1a}$$

and

$$b_{ij;k} = b_{ik;j} \qquad \text{Codazzi} \tag{1b}$$

where R_{ajqc} is the curvature tensor for R_n, $\varepsilon = \pm 1$ is the indicator for the normal to this n-space and; c denotes the covariant derivative.

In a usual embedding problem what is known is the R_n intrinsic geometry built from the metric tensor $g_{ab} = g_{ba}$, and the aim is to find out if the tensor b_{ac} satisfying (1a) and (1b) exists. For example, if $n = 4$ and the space-time has a vanishing Ricci tensor, then we know [1–3] that this b_{ac} does not exist and, hence, it may be concluded that no empty R_4 accepts embedding into E_5. Nevertheless, there are other metrics for which such tensor does exist [1, 4–7], and it is in these cases where a relation or identity to explicitly build the b_{ac} is needed.

In § 2 we will show that, using the Gauss equation (1a) only, it is possible to obtain and identity connecting b_{ac} with the intrinsic geometry of R_n, and we will indicate the conditions under which such an identity allows the explicit derivation of the second fundamental form. When $n = 4$ this identity reduces to a relation previously published by Goenner [8] as will be proved in § 3.

2. $\mathbf{b}_{ac}$ and the intrinsic geometry of $\mathbf{R}_n$

The Ricci tensor:

$$R_{jk} \equiv R^i_{jki} = \varepsilon(b^i{}_k b_{ji} - b b_{jk}), \quad b \equiv b^r{}_r \tag{2a}$$

and the scalar curvature:

$$R \equiv R^j{}_j = \varepsilon(b^{ac} b_{ac} - b^2) \tag{2b}$$

are obtained with the help of the Gauss equation. Then, (1a), (2a) and (2b) lead to the identity:

$$p b_{ac} = \frac{R}{2} R_{ac} + \frac{1}{2} R_a{}^r R_{rc} - R_{rc} - R_{arqc} R^{rq} - \frac{1}{4} R_{arqt} R_c{}^{rqt} \tag{3a}$$

that has not been found in the literature thus far. In this equation:

$$p \equiv b^{rq}\left(b^i{}_r b_{iq} - \frac{3}{2} b b_{rq} + \frac{b^2}{2} g_{rq}\right). \tag{3b}$$

But the Einstein tensor is given by:

$$G_{rq} = R_{rq} - \frac{R}{2} g_{rq} \tag{3c}$$

and, hence, (2a), (2b), (3b) and (3c) imply that

$$p = \varepsilon b^{rq} G_{rq} \tag{4a}$$

so that, after multiplying (3a) by εG^{ac} we obtain:

$$p^2 = \varepsilon\left(\frac{1}{2} R_a{}^r R_{rc} - R_{arqc} G^{rq} - \frac{1}{4} R_{arqt} R_c{}^{rqt}\right) G^{ac} \geqslant 0 \tag{4b}$$

which determines the value of ε when $p \neq 0$. From (4b) we learn that p is a function of the intrinsic geometry of R_n only.

Therefore, we can give to the embedding process the following sequence:

1) We have g_{ac} and do not know whether this R_n accepts a local and isometric embedding into E_{n+1}.
2) We compute p through (4b). Then we will have one of the two following cases:
a) If $p \neq 0$ then we determine b_{ac} with the help of (3a). Hence, the R_n does accept embedding into E_{n+1} if and only if *this* b_{ac} fulfils the Codazzi condition, eq. (1b).
b) When $p = 0$ eq. (3a) does not provide any explicit information about b_{ac}. Then, in order to decide if R_n is of class one, and to construct the corresponding second fundamental form, it is necessary to solve (1a) and (1b) directly.

3. The case $n = 4$

For ordinary space-time we can simplify (3a) and (4b) by making the dimension 4 more explicit. Indeed, recalling that the conformal tensor C_{arqt} and the double dual tensor $*R^*_{arqt}$ are related through the expression [9].

$$R_{arqt} = *R^*{}_{arqt} + 2C_{arqt} - \frac{R}{6}(g_{aq}g_{rt} - g_{at}g_{rq}) \tag{5a}$$

this can be introduced into the last term of (3a) to yield

$$pb_{ac} = \frac{1}{4}\left(\frac{K_2}{4} + R^2 - 3R_{qt}R^{qt}\right)g_{ac} - \frac{3}{2}C_{aqtc}R^{qt} + \frac{3}{2}\left(R_{aq} - \frac{R}{6}g_{aq}\right)R^q{}_c \tag{5b}$$

where Lanczos identity [10]

$$*R^{*a}{}_{cqt}R^{rcqt} = \frac{K}{4}g^{ar}, \quad K_2 \equiv *R^{*ijkr}R_{ijkr} \tag{5c}$$

has been used.

Equation (5b) is valid only for $n = 4$ and it is equivalent to (3.1) of Goenner (1976). Furthermore, (4b) reduces to

$$p^2 = -\frac{3\varepsilon}{2}\left(\frac{R}{24}K_2 + R_{iqtj}G^{ij}G^{qt}\right) \geqslant 0. \tag{5d}$$

As an application of (5b) and (5d) consider Gödel [11] cosmological model (signature $+2$):

$$ds^2 = -(dx^1)^2 - 2e^{x^4}dx^1dx^2 - \frac{1}{2}e^{2x^4}(dx^2)^2 + (dx^3)^2 + (dx^4)^2 \tag{6a}$$

which, together with (5d) imply that $\varepsilon = 1$ and $p = 3\sqrt{2}/4$. Therefore, from (5b) we get $b_{ac} = 0$, except

$$b_{11} = b_{44} = -\frac{\sqrt{2}}{2}, \; b_{12} = -\frac{\sqrt{2}}{2}, \; b_{22} = -\frac{3\sqrt{2}}{4}e^{2x^4} \tag{6b}$$

but this second fundamental form tensor does not satisfy (1b) since, for example, $b_{12;4} \neq b_{14;2}$. Then, due to remark 2 a) at the end of last section, we conclude that Gödel solution does not accept local and isometric embedding into E_5 as was already shown by Szekeres [2] using another technique. It is still ignored if (6a) admits embedding into E_6 (see [6]).

Notice that, when $p = 0$, it is not possible to use (5b) to construct b_{ij} and it is necessary to analyze the Gauss and Codazzi equations directly. An example of such cases can be found in Collinson [5], where Einstein-Maxwell metrics embedded into E_5 are studied.

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