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Recent developments in Mathematics and Quantum Field Theory

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1. Introduction

This talk is an assessment of the way Quantum Field Theory (QFT) has been developing in the last decade. QFT was originally formulated as a relativistic quantum theory of electrodynamics, and subsequently generalised to describe the weak and strong interactions as well. The formalism of QFT has applications in condensed-matter physics too. Attempts to construct a QFT for the gravitational interactions have been more or less unsuccessful so far. And for a long time, model QFT's with no direct physical application have been studied to understand their mathematical structure and physical properties. The goal of this exercise has been to gain intuition about QFT in general, by examining what are generally simplified situations – lower spacetime dimensions and fewer types of fields, for example.

In the last decade, more attention than ever before has been focussed on studying mathematical aspects of model QFT's. This development has reached the stage where a significant fraction of theoretical high-energy physicists are intensely involved in it, yet the experimental outcome of most of this work has been nil. Despite this fact, interest in this area is not flagging (if measured by number of papers published per year). The high-energy physics community has witnessed an unprecedented divorce between "theory" and "phenomenology", where the former has virtually no contact with experiment. It is this strange situation that will be examined in what follows.

2. Who ordered mathematical physics ?

When quantum mechanics was discovered, physicists had to learn mathematics which was not already familiar to them, such as linear algebra and infinite-dimensional vector spaces. But Dirac had this to say in the preface to his textbook [1], written in 1930: "From the mathematical side, the approach to the new theories presents no difficulties, as the mathematics required (at any rate that which is required for the development of physics up to the present) is not essentially different from what has been current for a considerable time." In other words, even if the mathematics was new for physicists, it was well-known to mathematicians. Thus, all that physicists had to do was read the relevant mathematics texts.

The quote above shows that Dirac realized how fortunate the situation was in his time, and that things might be different in the future. Indeed, in 1992, Dirac's statement would no longer be valid. The reason is that high-energy physicists are crucially interested in the grand unification of all fundamental forces. Since this includes gravity, and no ordinary QFT of gravity is known, we have been forced to turn to a promising new candidate: string theory. This is not a conventional QFT, but it is a theory (or class of theories) which postulates infinitely many fundamental particles, of which a finite subset are exactly massless and have gauge and gravitational interactions. Since string theories are generically renormalizable (even ultraviolet finite) quantum theories, and since they include gravity, they are the unique class of convincing candidates for grand-unified theories (GUTs) at present. In other words, at the present time high-energy physicists must either abandon grand-unification or study string theory.

Now for those who choose the second option, there is a crucial problem. String theory requires mathematics which is not generally known to physicists – but more than that, there is ample evidence that the relevant mathematical structures have not yet been discovered by humankind, and so one cannot learn them from textbooks. This means that string theorists have either to wait until mathematicians uncover these new structures, or else must throw themselves into the fray and become mathematicians for a while. The latter is what has happened. It is not yet the case that the mathematics discovered by string theorists (and other mathematicians working in similar areas) has found application to experimentally relevant physical problems, but the successes from the mathematical viewpoint have been remarkable. Indeed, this whole enterprise, started with a sound physical motivation (grand-unification) has produced evidence of a deeper unity of mathematical and physical ideas, whose full implications are far from clear at present.

It is illuminating to compare the situation in 1984, when string theory as a candidate GUT really captured the imagination of theoretical physicists, and 1954, when gauge theories were discovered. Yang and Mills [2] proposed $SU(2)$ gauge theory in 1954. Their physical motivation was to obtain a theory with local isospin invariance, since isospin symmetry between protons and neutrons was known, and the other known fundamental symmetry (based on electric charge) had been realized as a local symmetry in electrodynamics. Later, Utiyama [3] in 1956, and Glashow and Gell-Mann [4] in 1961, extended the theory of Yang and Mills to all simple Lie algebras. The latter work described various specific cases such as $SU(3)$, $Sp(2)$, G_2, and so on up to $O(9)$, in some detail. After this comes the observation "It is hard to imagine that any higher Lie algebras will be of physical interest". (This may be evidence that physicists then, as now, suffered from a limited imagination, but that is not the point here.) The important thing is that the entire set of simple Lie algebras was nicely classified in Cartan's book [5], so that physicists could directly use this classification to extract possible experimental consequences.

Yang-Mills theories really became relevant to physics after the discoveries of the Higgs mechanism, the confinement mechanism and the renormalizability of the theories. But the "static" problem of classification of possible gauge symmetries was solved, and physicists did not have to spend much time on that. They were free to address the "dynamic" problems, which require physical intuition and insight, and not sophisticated mathematics.

String theories as GUTs became popular in 1984 through a paper of Candelas, Horowitz, Strominger and Witten [6]. In particular, this paper proposed a some-what realistic compactification scheme for superstrings, based on three-complex-dimensional spaces called Calabi-Yau manifolds. But at the time, the classification problem for Calabi-Yau manifolds had scarcely been touched upon. Eventually, it was understood that Calabi-Yau compactifications were just a special class of two-dimensional Conformal Field Theory (CFT) [7] backgrounds for string propagation. The classification of CFT was an even bigger problem about which virtually nothing was known in 1984. Finally, it became clear that CFT's themselves should be thought of as classical solutions of string theory, while non-conformal two-dimensional QFT's represented "off-shell" configurations. This suggested the awesome prospect of classifying *all* two-dimensional QFT. Thus three classification problems, one nested inside the other, were unsolved in 1984 and remain far from solved today, despite enormous progress in the intervening period. The phenomenal complexity of these string-theory related mathematical problems contrasts sharply with the well-posed and completely solved problem of classifying all simple Lie algebras, which was useful for Yang-Mills theory.

3. Contributions of physics to mathematics

As a result of the difficult situation in which grand-unification finds itself, string theorists have been working on mathematical aspects of QFT. Because of the new and surprising insights into QFT provided by this kind of work, other theoretical physicists have also become involved in these mathematical problems, even if their own primary interest is not grand unification through strings. More surprisingly, because this activity has uncovered fertile areas in which to do mathematical research, mathematicians too have begun working on QFT.

Because physicists have been major participants in this game, a variety of techniques which are familiar to them have been used, even if they are not conventional in mathematics. This has provided an added power to the effort, and some of the results obtained would probably have been impossible to discover using only "standard" mathematical techniques. In Table 1 I give a brief and subjective list of the contributions that QFT has made to mathematics in the last decade. Most (but not all) of these are inspired in some way by string theory. I will comment on some unusual features of a few of these topics.

Some of these topics involve the use of QFT concepts to rederive known results in topology. In (1), for example, a class of supersymmetric quantum mechanics systems is constructed [8], which correspond to the motion of a particle on a given compact Riemannian manifold. Other degrees of freedom associated to complex structures or vector bundles on the manifold can be added. Then the vacuum functional integral for the system (with the right boundary conditions) is shown to be a topological invariant of the real or complex manifold, or the relevant bundle. On the one hand, this invariant can be associated to the index of a certain elliptic operator. On the other hand, it can be computed explicitly by using its topological invariance to reduce the functional integral to an ordinary integral over constant configurations [9,10]. Thus one obtains various index theorems for the corresponding elliptic operators. This was subsequently generalized to trace theorems on non-compact spaces [11].

One outgrowth of this work which could have genuine physical implications was

Table 1.

	Physics		Mathematics
1	Supersymmetric Functional Integral	$\rightarrow$	Index Theorems
2	Conformal Field Theory	$\rightarrow$	Θ-function Identities, Fuchsian Differential Equations, Verlinde's Theorem
3	Integrable Lattice Models	$\rightarrow$	Quantum Groups
4	Chern-Simons Gauge Theory	$\rightarrow$	Invariant Polynomials for Knots and Links
5	Topological Gauge Theory in Four Dimensions	$\rightarrow$	Donaldson Polynomials for 4-Manifolds
6	$N = 2$ Superconformal Field Theory	$\rightarrow$	Mirror Manifolds
7	$c < 1$ String Theory	$\rightarrow$	Intersection Theory on Moduli Space, KdV Hierarchy
8	$c = 1$ String Theory	$\rightarrow$	Euler Character of Moduli Space

the discovery that in some compactified superstring theories, the number of chirally coupled fermion species is governed by topological indices of the compactification manifold [12], such as its Euler characteristic. The existence of several fermion generations (identical except for their masses), a long-standing puzzle in fundamental physics, would thus be explained in terms of topology.

In some cases, the QFT derivation of mathematical results, even if less rigorous, is more powerful and can provide generalizations of the existing results. In (4) and (5), certain polynomial invariants were obtained as correlation functions of operators in suitably chosen QFTs. In (4), the theory is a topological theory of gauge fields alone, in three dimensions [13]. The correlators of Wilson loops running over knots or links in the chosen 3-manifold are polynomial invariants of the associated knots and links, reproducing in particular the famous Jones polynomials [14]. But in this class of theories, one can choose various 3-manifolds rather than just the 3-sphere, and some information about conformal field theory enables one to obtain link invariants for these 3-manifolds, generalizing the work of Jones. QFT appears to provide such a natural setting for such problems that generalizations and heuristic derivations become rather easy.

In the study of conformal field theories (CFT) on general compact Riemann surfaces, it was shown that one can compute correlation functions of free fermions in terms of higher-genus theta functions. This computation follows from Wick's theorem for free fermions. On the other hand, one can bosonize the free fermions into vertex operators of a free scalar field, in which case different free-field techniques can be used, to obtain distinct expressions for the correlators, again in terms of theta functions. The statement of boson-fermion equivalence then becomes an identity among different theta-functions on higher-genus Riemann surfaces [15-17]. This approach reproduces in particular Fay's trisecant identity [18].

A remarkable class of mathematical results which was previously unknown and even unsuspected, emerged from the study of a class of $N = 2$ supersymmetric CFT. It is known that there is such a CFT for every Calabi-Yau manifold [6]. (A Calabi-Yau manifold is a complex 3-manifold with vanishing first Chern class.) Moreover, if this CFT is used as a background for superstrings to propagate in, then roughly half the couplings of the effective field theory can be computed exactly as topological invariants of the Calabi-Yau manifold. Astonishingly, it turns out that in many cases there are two completely distinct Calabi-Yau manifolds describing the same CFT (see [19] for details). The two manifolds (called a "mirror pair") have the property that while half the couplings of the effective field theory are computable from the topology of one manifold, the other half of the couplings are computable from the other. Thus a mirror pair represents an exactly solvable background for superstring propagation. And, most interesting for mathematicians, the mirror phenomenon links apparently very distinct manifolds via the fact that they describe the same CFT. (A very simple version of the mirror phenomenon appears in the famous "duality" of string theory compactified on a torus: the torus and its dual torus describe the same string theory.)

The study of integrable lattice models is rather old, but of late the mathematical structures appearing in the Vertex Models and Restricted Solid-on-Solid (RSOS) Models [20,21] began to be understood. These structures are generalisations of Lie algebras, called Quantized Universal Enveloping Algebras, or for short, Quantum Groups [22]. The study of these algebras, their representations and their relation with braid groups and CFT has been a major activity over the last few years, and much of this work is again due to physicists who were familiar with QFTs or at least their discretized lattice versions.

In the next section I examine cases (7) and (8) above, related to non-critical string theory, in some more detail.

4. Non-critical string theory and the moduli space of Riemann surfaces

A new formulation of two-dimensional gravity was studied intensely from late 1989 [23]. In this formulation, the problem is studied in terms of statistical sums over discrete random surfaces. These sums can be formulated as functional integrals over random matrix-valued fields. The simplest case, that of a single constant matrix variable, was found to have interesting multicritical points where the large-N limit, taken in a particular way, leads to a well-defined continuum theory. The result is a non-linear differential equation for the partition function of two-dimensional gravity coupled to certain conformal field theories. This model was generalized to the case of many constant matrices. Perturbations of these theories are described by the integrable hierarchy of KdV equations and their generalizations.

Subsequently, this work was re-interpreted in the language of intersection numbers of cohomology classes on the moduli space of Riemann surfaces [24]. Correlation functions of matrix variables in these theories give the intersection numbers in question. Thus the study of random-matrix integrals (which have been studied long ago in connection with the large-N expansion of gauge theories) gave an insight into the computation of certain topological indices, the intersection numbers, for the moduli space of Riemann surfaces. Even more, it showed for the first time that this topological problem is governed by the integrable KdV flow equations [25].

Another matrix model that was studied involves a single random matrix variable which now depends on one continuous parameter [26]. Its physical interpretation was understood to be of a string moving in a specific two-dimensional spacetime [27]. The partition function of this theory in genus g, in the case where the continuous variable takes values on a circle of a specific radius, is given by [28]

$$Z_g = (-1)^g \frac{1}{2g(2g-2)} B_{2g},$$

where B_n is the nth Bernoulli number. Interestingly, it is known from works in the mathematical literature [29,30] that this number is precisely the (virtual) Euler characteristic of the moduli space of genus-g Riemann surfaces with no punctures. Thus, there exists a string-theory background in which the partition function is an important topological index associated to Riemann surfaces. This is another instance where a physical problem in QFT led to a result of mathematical significance. This result also led to a recent work in which a topological QFT description of strings propagating in two-dimensional spacetime was found [31].

5. Conclusions

We learn from all this that the study of aspects of QFT (often related to string theory) has led to a large number of mathematical results in topology and algebraic geometry. In some cases, known results can be speedily recovered, in other cases they can be generalized as well.

It is clear that in the next few years, more input of this kind from QFT to mathematics will be discovered. It is not unreasonable to believe that QFT itself is an important branch of mathematics, which plays a crucial role in unifying and explaining several diverse mathematical results.

On the other hand, the input from these and other mathematical results to experimentally accessible physics problems has not been very significant so far. It is probable that in the near future, we will see the last decade of mathematically oriented research in QFT begin to bear fruit for the physics problems of interest, most of all for the problem of finding a grand-unified theory of fundamental interactions including gravity.

In 1931, Dirac said [32]: "Advance in physics is to be associated with a continual modification and generalization of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation". In 1992, this is more true than ever before.

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