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Two-dimensional gravity, matrix models and nonperturbative string theory

GAUTAM MANDAL

Tata Institute of Fundamental Research, Bombay 400005, India

Abstract. We review some of the recent developments in nonperturbative string theory and discuss their connections with black hole physics and low dimensional fermi systems.

1. Introduction

In this section I shall briefly motivate the need to consider nonperturbative string theory. Perturbative string theory, as defined by a sum over two-dimensional surfaces of different genera (number of handles) [1] provides the first ever (and so far the only) example of a theory of interacting gravitons that is equivalent to Einstein's theory at low energies [2] and is finite in perturbation theory under appropriate circumstances. While this is an important step towards finding a quantum theory of gravity, there are several questions that remain unanswered in the perturbative approach. To pick a few:
(a) given the fact that perturbative string theory is defined necessarily in a fixed background geometry, is there a nonperturbative formulation (lagrangian or otherwise) that can determine the background geometry itself?
(b) if the background geometry involves strong curvature then one expects the coupling constant to grow large which makes perturbation theory in such backgrounds unreliable; how does one define the theory nonperturbatively in such cases?
(c) a related question is: in the regions of large curvature since string theory becomes strongly coupled, the notion of an expectation value of the metric itself becomes unreliable thanks to strong quantum fluctuations of the metric and its strong mixing with the higher string modes; clearly the picture of "spacetime" needs a revision — what is the new picture?

Besides these there are questions related to non-gravity aspects of string theory which also require a nonperturbative definition. For instance, one needs to have a nonperturbative mechanism of supersymmetry breaking because breaking SUSY at any finite order necessarily spoils the finiteness of string theory (at least till date). There is also this interesting observation [3] that string perturbation theory, even when finite, is non-Borel summable in a certain way, one interpretation of which is the existence of instanton contributions that are stronger than in conventional field theories and which go as $\exp(-1/g_{string})$. A nonperturbative formulation is clearly tailored to probe such effects.

The only workable nonperturbative string theory at the moment is defined in terms of the so-called "matrix models". In the rest of the talk I will define and describe matrix models and discuss how they address some of the questions listed above.

2. Matrix models and dynamically triangulated random surfaces

Random matrices have been used in nuclear physics for a long time [4] largely because of the connection between nuclear spectral density and eigenvalue density of random matrices. The connection between functional integral over matrix-valued fields and two-dimensional surfaces of different topologies was first made in the path-breaking papers [5] by 't Hooft on dual string models and large-N QCD. To understand this connection let us first discuss the discrete (lattice) approach to quantum gravity.

The subject of quantum gravity in the continuum is fraught with several basic questions of principle. Quite besides the issue of nonrenormalizability which is the major problem is four dimensions, there is the issue of the metric-dependence of the short-distance cut-off (needed to ensure general coordinate invariance in the quantum theory) which makes even the *definition* of the theory problematic in all dimensions. One of the early remedies considered was to think of lattice gravity or "dynamically triangulated random surfaces" (DTRS). As in the case of lattice gauge theories which are automatically gauge invariant, the DTRS formulation is also *a priori* general coordinate invariant. In two dimensions, the original ideas of Regge [6] were adopted in the following form. The lattice regularization of Polyakov path integral [1]

$$Z = \sum_h \int \mathcal{D}g_{ab}(\xi)\mathcal{D}X^\mu(\xi)\exp[-\int d^2\xi\sqrt{g}\{\beta + \frac{\beta'}{4\pi}R + \frac{1}{2}g^{ab}\partial_a X^\mu \partial_b X^\mu\}] \quad (1)$$

is taken to be [7]

$$\begin{aligned} Z &= \sum_G \Gamma(G)e^{-\beta p(G)-\beta' h(G)}, \\ \Gamma(G) &= \int \prod_i \frac{d^D\mathbf{x}_i}{\pi^{D/2}} \exp[-\frac{1}{2}\sum_{\langle ij\rangle}(\mathbf{x}_i - \mathbf{x}_j)^2]. \end{aligned} \quad (2)$$

In the first equation there is an explicit summation over genus h; since $(4\pi)^{-1}\int\sqrt{g}R = (2-2h)$ one could pull out the β' term outside to display the sum over genus as $\sum_h \exp[-\beta'(2-2h)]$. In the second equation the sum is over all possible triangulations G. We denote the number of links (edges) of G as $l(G)$, the number of sites (vertices) as $s(G)$ and the number of plaquettes (faces or triangles) as $p(G)$. The notation $h(G)$ stands for the genus of the simplex G, defined by the Euler formula

$$h(G) = s(G) - l(G) + p(G). \quad (3)$$

The connection with matrix models comes about as follows. Consider the following partition function

$$Z(g) = \int dM\exp[-\mathrm{tr}V(M)], \quad V(M) = (\frac{1}{2}M^2 + gM^3). \quad (4)$$

This is the simplest kind of matrix model, with M an $N \times N$ hermitian matrix. As we shall see, this will correspond to pure gravity in two dimensions. More generally one can consider $M(\mathbf{x})$, which will typically correspond to two-dimensional QG coupled to D scalar fields as in (1). One can also consider matrices which are unitary or orthogonal. We will discuss these cases later.

Now consider expanding (4) in powers of g:

$$\begin{aligned} Z(g) &= Z(0)[\sum_n a(n) g^n], \\ a(n) &= \frac{\int dM e^{-\mathrm{tr}\, V(M)} (\mathrm{tr}\, M^3)^n}{\int dM e^{-\mathrm{tr}\, V(M)}}. \end{aligned} \tag{5}$$

Like in standard field theory $a(n)$ can be computed by means of Feynman diagrams, using the Wick contraction

$$\langle M^*_{ij} M_{kl} \rangle_0 \equiv \frac{1}{Z(0)} \int dM M^*_{ij} M_{kl} e^{-\mathrm{tr}\, M^2/2} = \delta_{ik}\delta_{jl}. \tag{6}$$

The double Kronecker delta can be kept track of by using a pair of oppositely directed lines ($\rightleftharpoons$) [5]. An analogous physical picture is that the propagator for mesons, viewed as a quark-antiquark combination, can be viewed as a pair of oppositely directed quark propagators. Since the matrix propagator (6) is similar to the meson propagator, the counting problems are also similar. The Feynman diagrams in $a(n)$ are all possible graphs one can draw with n vertices while ensuring continutity of each quark line. An important rule that emerges is that each closed quark loop yields a factor N, representing summation over the "colour" index. It was shown by 'tHooft that if one redefines the coupling constant $g = \bar{g}/N$, then the only N-dependence of a Feynman graph $\widetilde{G}$ appearing in $a(n)$ comes from its genus $h(\widetilde{G}) \equiv [s(\widetilde{G}) - l(\widetilde{G}) + p(\widetilde{G})]/2$, in the form $N^{2-2h(\widetilde{G})}$. Thus, we have the following result

$$a(n) = \sum_{\widetilde{G}}^{(n)} N^{2-2h(\widetilde{G})}, \tag{7}$$

where $\sum_{\widetilde{G}}^{(n)}$ is over all Feynman graphs with $s(\widetilde{G}) = n$. This gives

$$Z = \sum_{\widetilde{G}} \bar{g}^{\,s(\widetilde{G})} N^{2-2h(\widetilde{G})}. \tag{8}$$

Now compare this expression with

$$Z = \sum_G \exp[-\beta p(G) - \beta' h(G)] \tag{9}$$

which is simply (2) evaluated in $D = 0$ so that $\Gamma(G) = 1$. Indeed, if we define the graph $\widetilde{G}$ to be the dual to G, so that $l(G) = l(\widetilde{G}), s(G) = p(\widetilde{G})$ and $p(G) = s(\widetilde{G})$ and make the identification $\bar{g} = \exp[-\beta]$ and $N = \exp[-\beta']$ then one can easily see that summands are the same in both (8) and (9). It can also be shown that the two sets, $\{G\}$ of triangulations and $\{\widetilde{G}\}$ of Feynman graphs, are one-to-one under the duality transformation. Thus we have (8) = (9).

How about higher D? The analysis is pretty similar [8] and the final result is that (2) is equal to the matrix model

$$Z = \int \prod_{\mathbf{x}} dM(\mathbf{x}) \exp[- \int d^D\mathbf{x} \mathrm{tr}\left[M(\mathbf{x})e^{-\partial_{\mathbf{x}}^2/2}M(\mathbf{x}) + V(M(\mathbf{x}))\right]. \tag{10}$$

Continuum Limit: The lattice regularization (2) is of course a poor approximation to (1) unless one can show that the sum $\sum_G$ in (2) is dominated by contributions of those triangulations which have $p(G) \to \infty$. In particular

$$\langle p(G) \rangle = \partial \ln Z(\beta)/\partial\beta \tag{11}$$

must diverge. This is achieved by carefully adjusting the value of β so that the perturbation expansion in β diverges. We call such a value of β its critical calue, β_c, and give the name "continuum limit" to the limit

$$\beta \to \beta_c. \tag{12}$$

Large N limit: Expressions like (8) have another obvious limit, $N \to \infty$, in which the only graphs that survive are the "planar" ($h = 0$) ones. 'tHooft's original idea was to study QCD in the large N limit where one can ignore the non-planar diagrams.

Double scaling limit: [9-11] This is the limit in which one simultaneously takes the two limits described above in such a way that the non-planar diagrams survive and one achieves contiuum limit for all genera. Intuitively this works because the critical value of β or equivalently the critical value of the cosmological constant is related to the local properties of the two-dimensional surface and is therefore genus independent (the corresponding statement in statistical mechanics is that the critical temperature of a system depends essentially on the bulk properties). This implies that if we make a genus-expansion of Z in quantities like (10)

$$Z_N(\beta) = \sum_h N^{2(1-h)} F_h(\beta), \tag{13}$$

then $F_h(\beta)$ goes critical for each h at the same value of $\beta = \beta_c$ (for a cubic potential in (10), β is to be identified with the cubic coupling as explained before; for more generic potentials one can identify the overall scale of the potential with β modulo suitable renormalization of the quadratic term [11]). Indeed it can be shown, using matrix model methods [12] as well as continuum methods [13] that the precise behaviour of $F_h(\beta)$ is

$$F_h(\beta) \propto (\beta - \beta_c)^{(1-h)(\gamma-2)},$$

where γ is the socalled "string susceptibility". The case $h = 1$ (torus) is exceptional where $F_1 \propto \ln(\beta - \beta_c)$. This enables us to take the limits $N \to \infty$, $\beta \to \beta_c$ in such a way that

$$g_{string}^2 \equiv \left(N^2(\beta - \beta_c)^{2-\gamma}\right)^{-1}. \tag{14}$$

Note that with this the matrix model partition function automatically arranges itself so that the genus h contribution is weighted by a factor g_{string}^{2h}, as in standard string perturbation theory.

The important achievement of matrix models is that it can evaluate the original partition function without recourse to the genus expansion.

3. Multicriticality and $c < 1$ models

So far we have talked about cubic potentials $V(M)$. What happens if one generalizes to arbitrary polynomials? Kazakov [14] showed that something analogous to Landau-Ginzburg description of multicritical behaviour takes place. Namely, for a generic potential the universal behaviour is the same as that of the cubic potential and the string exponent is given by $\gamma = -1/2$. However, if one considers higher polynomials and fine tunes m of its parameters suitably, then the universality class changes to a higher multicritical point, characterized by $\gamma = -1/k$. We have seen that $k = 2$ corresponds to pure $D = 0$ gravity. It can be shown that the general k-th multicritical point correponds to conformal matter labelled by the Kac indices (p, q) where $p = 2k - 1$ and $q = 2$, coupled to two-dimensional gravity.

The physics of these models is explored by the method of orthogonal polynomials [15] which leads to differential equations satisfied by the "specific heat" (second derivative of partition function with respect to the cosmological constant). For the one-matrix models described above these differential equations are the Painleve transcendents. The emergence of these differential equations in one-matrix models led to the construction of the general $c < 1$ matrix model by Douglas [16] where he showed that by considering *two-matrix models* and fine tuning its parameters one can describe all $c < 1$ conformal matter systems coupled to two-dimensional gravity.

There are several interesting physics calculations that one can do using this technology. First of all, one can match the matrix model partition functions with those calculated in the continuum theory. The latter are available only at low genus. There is complete agreement for all the models described above. The advantage of the matrix model over the continuum methods is already clear since it gives us the value of the partition functions at arbitrary genus and indeed even nonperturbatively. One can also calculate correlation functions. Matrix monomials in the double scaling limit correspond to vertex operators in the continuum model. Correlation functions of these also match at low genus with continuum results [17] and generalize them nonperturbatively.

There are a number of interesting physics results that emerge from the study of the $c < 1$ models. One of them is the issue of large orders of perturbation theory. By studying the high genus behaviour of the partition function or the correlation functions, one sees that the perturbation series is actually divergent. Shenker [3] made the observation that such divergence is characteristic of *string theory* itself and had earlier showed up in critical string theory as well. He suggested that it indicates existence of nonperturbative effects which go as $\exp(-1/g_{string})$.

Very interestingly, this issue is closely related to another important issue [18-22] which appears in the $c < 1$ models. Note that a cubic potential for the matrix model is unbounded and hence the partition function for pure gravity is ill-defined. One way to make this problem well-defined is by adding a quartic term of the right sign in the potential which makes it bounded. This leads to an asymmetric double well potential. It can be shown that in the double scaling limit eigenvalues tunnel across from the metastable well to the global minimum in finite time. The tunnelling amplitude can be calculated and it is given by [20,22] $\exp(-1/g_{string})$. Thus, Shenker's nonperturbative effect is connected with instability and tunnelling. In the Marinari-Parisi model [18] this is connected with supersymmetry breaking.

Indeed this gives the first dynamical mechanism of nonperturbative supersymmetry breaking, thus answering one of the questions raised in the introduction. To understand this effect in a field theory framework, see [23].

4. c=1 model

By far the most interesting matrix model is the one that corresponds to $c = 1$ matter coupled to gravity. As discussed in the last section, the partition function is obtained by considering (10) for $D = 1$. It can be argued that the quadratic term $\mathrm{tr}\left[M(x)\exp(-\partial_x^2/2)M(x)\right]$ is in the same university class as $\mathrm{tr}\left[M(x)(1-\frac{1}{2}\partial_x^2)M(x)\right]$. The last form has a long and beautiful history. It was shown in [24] that this problem is exactly mapped onto a fermion problem in one-dimension moving in an external potential. Essentially the van der Monde determinant coming from integrating out the "angle" variables makes the eigenvalues behave as fermions. The double scaled theory was constructed in [25]. It was recognized in [26] that this is equivalent to a nonrelativistic fermion field theory and this observation was used to derive the low enrgy (relativistic) excitations and interactions. The physics of low energy particle-hole pairs is identifiable with that of the $c = 1$ tachyon which can also be described by a collective field theory [27]. There is remarkable agreement between scattering matrices of tachyons calculated using matrix model methods [26,28-31] and continuum methods at low genus [32].

There is however a lot of interesting physics in the $c = 1$ model that comes from beyond the low energy physics of the tachyon and can be extracted by observing [33-35] that the full non-relativistic fermion theory has a symmetry called the W_∞ symmetry. This symmetry has been observed [36] in the continuum theory as the symmetry that relates the "discrete states" of the $c = 1$ model which are remnants of the higher string modes. Using W_∞ symmetry one can [35] write down an exact bosonic field theory that captures the physics of the tachyon and the higher modes. This provides us the only non-trivial nonperturbatively solvable string field theory so far.

Black holes: Another interesting feature of $c = 1$ continuum theory coupled to gravity or equivalently [37-40] two-dimensional string theory is the existence of the two-dimensional black hole [41,42]. There has been a long desire to find a black hole solution in string theory to ask if the black hole singularity can exist in a consistent theory of quantum gravity. This question of course cannot be settled in perturbation theory because the coupling constant grows large near the singularity leading to a meaningless perturbation expansion. Fortunately it has been possible recently to find the black hole solution in the nonperturbative framework of $c = 1$ matrix model [43] (see also [44-47]) so that one can address the question of black hole singularity meaningfully. The answer is [43,48] that in the exact quantum theory the black hole singulrity indeed disappears! Indeed the black hole singularity can be identified with singulrities associated with divergences of the semi-classical expansion. It would be very interesting to see if this feature persists for the four-dimensional black hole and other cosmological singularities, *e.g* the initial singularity.

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