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Construction of topological conformal field theories

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Abstract. In this lecture I review the construction of two dimensional Topologiccal Conformal Field Theories from $N = 2$ superconformal theories. We show that a BRST structure emerges upon a twisting of the $N = 2$ superconformal algebra. Moreover, the energy-momentum tensor of the twisted theory is BRST-exact and all the physical correlation functions are independeent of the two dimensional metric. We briefly mention several generalizations such as the construction of topological superconformal theories as well as the topological conformal theories on higher genus Riemann surfaces.

Topological field theories [1,2,3] (TFT) have attracted considerable attention recently due to their rich mathematical structure as well as strong physical motivations. It has been conjectured that TFT's may be related to the ordinary field theories via phase transitions. Their study may be important in understanding the nature of these phase transitions. It has also been shown that TFT's in two dimensions have intimate connections with the quantum gravity [4,5] and matrix models [6,7]. TFT's are characterized by the independence of the physical correlation functions from the space-time metric or the local coordinates, although the field theory action itself is in general metric dependent.

The topological structure in a two dimensional theory was first shown by Witten [1]. He showd that a topological field theory can be obtained from an N=2 supersymmetric sigma model. Eguchi and Yang [8] extended his work to the construction of topological conformal field theory (TCFT). They showed that by a suitable modification of the energy-momentum tensor in an N=2 superconformal theory a topological structure emerges and all the correlation functions become independent of the local coordinates. Such topological theories have also been coupled to the toplogical gravity [1,2,3,9] and connection of the combined theory with multi-matrix models [10,11] have been shown. The N=2 construction of TCFT and its coupling to the topological gravity has been explicitly worked out in [11].

We now describe the construction of these thoeries from $N = 2$ superconformal theory [8] in two dimensions. We first obtain the symmetry algebra of the TCFT and later on discuss the Hilbert space structure. The symmetry generators of the $N = 2$ superconformal theories are, the Conformal generators L_m, two superconformal generators $G_r^{\pm}$ and a $U(1)$ current J_m. The symmetry algebra can be written as,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{1}$$

$$\begin{aligned}
[L_m, G_r^{\pm}] &= (\frac{m}{2} - r)G_{m+r}^{\pm}, && (2)\\
[L_m, J_n] &= -nJ_{m+n}, && (3)\\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n}, && (4)\\
[J_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm}; && (5)\\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, && (6)
\end{aligned}$$

where indices $m, n, ...$ and $r, s, ...$ are intergral and half-integral respectively. The algebraic structure of a TCFT emerges after a twisting of the energy-momentum tensor of the thoery as,

$$T(z) \rightarrow T(z) + \frac{1}{2}\partial J(z) \tag{7}$$

which, in view of the mode expansion of a conformal field ϕ with weight Δ:

$$\phi^{\Delta} \equiv \Sigma_m \phi^{\Delta}{}_m z^{-m-\Delta}, \tag{8}$$

implies

$$L_m \rightarrow L_m' = L_m - \frac{1}{2}(m+1)J_m, \tag{9}$$

where

$$\begin{aligned}
T(z) &= \Sigma_m L_m z^{-m-2}, && (10)\\
J(z) &= \Sigma_m J_m z^{-m-1}. && (11)
\end{aligned}$$

It can now be verified that L_m' satisfy the conformal algebra without a central term:

$$\left[L_m', L_n'\right] = (m-n)L_{m+n}'. \tag{12}$$

The absence of the central term in the Virasoro algebra (12) is a hint that the twisted theory is topological, since the two as well as higher point functions of the energy-momentum tensor vanish. For example,

$$< T(z_1)T(z_2) > \sim \frac{c}{2}(z_1 - z_2)^{-4} = 0\ . \tag{13}$$

To prove the topological nature of the theory more rigorously, we first show the existence of a BRST charge in the theory. For this, we notice that the modified conformal wights of the operators G^+ and G^- with respect to the new Virasoro generator L_m' are 1 and 2 respectively which can be seen from the commutators:

$$\begin{aligned}
\left[L_m', G_r^+\right] &= -(r + \frac{1}{2})G_{m+r}^+, && (14)\\
\left[L_m', G_r^-\right] &= -[m - (r - \frac{1}{2})]G_{m+r}^-. && (15)
\end{aligned}$$

Due to the change in the conformal weights of these generators and in view of the mode expansion (8) we redefine,

$$\begin{aligned}
Q_n &\equiv G_{r+\frac{1}{2}}^+, && (16)\\
G_n &\equiv G_{r-\frac{1}{2}}^-, && (17)
\end{aligned}$$

and obtain

$$\left[L'_m, Q_n\right] = -nQ_{m+n}, \tag{18}$$
$$\left[L'_m, G_n\right] = (m-n)G_{m+n}. \tag{19}$$

In particular, due to the redefinition in eqn.(16), Q_0 is simply $G^+_{-\frac{1}{2}}$. In terms of these redefined operators, the rest of the algebra can be written as

$$[J_m, J_n] = \frac{c}{3}m\delta_{m+n}, \tag{20}$$
$$[J_m, G_n] = -G_{m+n}, \tag{21}$$
$$[J_m, Q_n] = Q_{m+n} \tag{22}$$
$$\left[L'_m, J_n\right] = -nJ_{m+n} + \frac{c}{6}m(m+1)\delta_{m+n,0}, \tag{23}$$
$$\{G_m, G_n\} = \{Q_m, Q_n\} = 0, \tag{24}$$
$$\{Q_n, G_m\} = 2L'_{m+n} + 2nJ_{m+n} + \frac{c}{3}n(n-1)\delta_{m+n,0}. \tag{25}$$

From the above commutation relations we notice the presence of a nilpotent operator Q_0 in the modified theory. Moreover, since $Q(z)$ has weight 1, Q_0 is interpreted as a BRST charge. The modified energy-momentum tensor is a BRST exact operator which can be seen from the commutation relation:

$$\{Q_0, G_m\} = 2L'_m. \tag{26}$$

Now, to show that the correlation functions of the twisted theory are independent of the two dimensional space-time coordinates, we restrict the physical Hilbert space to those states which are annihilated by the BRST operator Q_0, i.e.

$$Q|\phi >= 0, \tag{27}$$

which implies that the physical operators commute with the BRST charge. Since L'_{-1} is the generator of translation, the n-point function of n such operators satisfy the condition,

$$\partial_{z_i} < \phi_{(z_1)}, ..., \phi_{(z_n)} >=< ..[L'_{-1}, \phi_{(z_i)}]... > . \tag{28}$$

Then using the BRST exactness of L'_{-1} and the fact that Q commutes with all the physical operators and annihilates the vacuum we get the condition,

$$\partial_{z_i} < \phi_1, ..., \phi_n >= 0, \tag{29}$$

which implies the z_i indepence of the correlators. Therefore we have proved that any physical correlation function in the TCFT obtained by the twisting, eqns. (7) and (9), of the $N=2$ superconformal theory is independent of the coordinates z_i.

In view of the results of the previous paragraph, we interpret the commutation relations in eqns.(12) and (18) - (25) as the algebraic structure of a topological conformal field theory. The oprator J is interpreted as a ghost charge. From eq.(21)(22), we get the ghost charges for G and Q to be -1 and $+1$ respectively.

We now obtain the Hilbert space of the topological conformal theory from the knowledge of the Hilbert space of the $N=2$ superconformal theory. For $N=2$ theory it has been proved earlier [12] that any general state can be decomposed as

$$|\phi>=|\phi_0>+G^+_{-\frac{1}{2}}|\phi_1>+G^-_{+\frac{1}{2}}|\phi_2>, \tag{30}$$

where $|\phi_0>$ is a "chiral-primary" field defined by the chirality condition,

$$G^+_{-\frac{1}{2}}|\phi_0>=0 \tag{31}$$

and the "primary field condition",

$$G^+_r|\phi_0>=G^-_r|\phi_0>=0, \quad r>0. \tag{32}$$

The physical Hilbert space of the twisted theory is defined by the BRST cohomology condition (27), which by the identification of Q as $G^+_{-\frac{1}{2}}$ is simply the chirality condition eqn.(31) in the untwisted theory. We therefore conclude that the Hilbert space of the twisted theory is given simply by the chiral fields of the original $N=2$ theory.

The chiral field decomposition can be obtained from eqn.(30) and is written as

$$|\phi>=|\phi_0>+G^+_{-\frac{1}{2}}|\phi_1>. \tag{33}$$

For the twisted theory, we therefore have

$$|\phi>=|\phi_0>+Q|\phi_1>, \tag{34}$$

which implies that out of the set of chiral fields, only those which are "chiral-primary" are nontrivial. Rest of the fields are BRST exact.

The conformal weights of the physical states in the topological theory are obtained by using the condition

$$h=\frac{q}{2} \tag{35}$$

for the chiral states of the $N=2$ superconformal theory [12] where h and q are the conformal weights and the $U(1)$ charges for these states in the untwisted theory. Then, since

$$L_0^{'}=L_0-\frac{1}{2}J_0, \tag{36}$$

the modified conformal weight $h^{'}=0$. The spctrum of the TCFT therefore contains only a set of weight zero fields which in the untwisted theory are simply the chiral primary fields.

A known class of TCFT are the Topological minimal models, constructed from $N=2$ superconformal minimal models. In terms of a label K parameterizing the central charge of the $N=2$ superconformal field theory as $c=\frac{3K}{K+2}$, the topological minimal models have $K+1$ physical fields in its Hilbert space. The corresponding operators form a ring structure and all the correlation functions can be determined in terms of the ring structure constants. When copuled to the two dimensional topological gravity, these minimal models give an alternative representation for the coupling of the two dimensional quantum gravity with the $c<1$ Virasoro minimal models.

The constructions of the TCFT's have been generalized in several directions. For example, present author and collaborators have analyzed whether additional symmetry structures are allowed together with the topological structure. It was found [13] that by starting from $N = 3$ and $N = 4$ [14,15] superconformal thoeries, instead of the $N = 2$ theories, twistings can be done in such a manner that the twisted theory has additional superconformal symmetry in addition to the BRST symmetry. Hence, these theories can be interpreted as Topological superconformal thoeries.

In another generalization, the construction of TCFT's on higher genus Riemann surfaces was also analyzed by the author and collaborators [16]. The starting theory for the higher genus surfaces are characterized by the Krichever-Novikov algebra which is a generalization of the superconformal algebra and reduces to it on the sphere. We showed that the $N = 2, 3, 4$ Krichever-Novikov algebra can be consistently twisted so as to give a BRST structure on the higher genus Riemann surfaces.

Finally, I would like to remark that there are still many unexplored topics in the context of TCFT's. For example, it is worth examining if one can couple the topological matter theories described above with the usual string coordinates to construct new string theories. Since the TCFT's have $c = 0$, this is certainly allowed by the central charge condition $c = 26$. Such investigation may open the way for another class of string model building.

Acknowledgements

This lecture is a review of the work in refernces [8,13,16].

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Discussion

Avinash Khare : Could you make a few remarks on the Charm-Simons type of topological field theories?

Alok Kumar : A natural question to ask is whether the Charm-Simons type of topological field theories are twisted N=2 supersymmetric theories? The answer, I think, is NO. Moreover in Charm-Simons theories, the action itself is topological unlike the topological theories we discussed, where the action was metric dependent.

Probir Roy : Are there equivalents of topological field theory in statistical mechanics, say spin systems?

Alok Kumar : I remarked in the lecture that topological conformal field theories describe two dimensional minimal models coupled to the quantum gravity in 2-dimensions. From a lattice theory point of view, there are statistical mechanical models on a random lattice, From this point of view, for example, the Ising model on a random lattice can be described as topological field theory.

U. Sarkar : Is there any correspondence between the topological $N = 2$ supercomformal field theory and the geometric compactification of string theory in the point field theory limit.

Alok Kumar : Topological Landau-Ginsburg models exist. But since they are a central charge $c = 0$ system, they cannot compensate for the extra dimensions. It is therefore unlikely that they can be used for a geometric compactification like in the Calabi-Yan scheme in a conventional way.