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An introduction to anyons

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1. Introduction

In three space dimensions, there are only two kinds of statistics available to indistinguishable quantum particles — bose statistics or fermi statistics. Namely, if $\psi(\mathbf{r}_1\ \mathbf{r}_2\ ...\ \mathbf{r}_N\ )$ is a N-particle wave function, then

$$\psi(\mathbf{r}_1\ \mathbf{r}_2\ ..\ \mathbf{r}_i\ ..\ \mathbf{r}_j\ ..\ \mathbf{r}_N) \ = \ +1 \text{ or } -1 \text{ times } \psi(\mathbf{r}_1\ \mathbf{r}_2\ ..\ \mathbf{r}_j\ ..\ \mathbf{r}_i\ ..\ \mathbf{r}_N)$$

No 'intermediate' statistics is available in three dimensions if we restrict ourselves to scalar (single-component) wave functions.

In one or two dimensions, however, a continuous range of statistics is available. A given system of indistinguishable particles may be labelled by a statistics parameter ν such that if $\nu = 0$, the particles are bosons, whereas if $\nu = 1$, the particles are fermions.

The way this works in two dimensions is as follows. Suppose that the wave function $\psi(\mathbf{r}_1\ \mathbf{r}_2\ ...\ \mathbf{r}_N)$ is defined only if $|\mathbf{r}_i - \mathbf{r}_j| > a$ (this is called the hard core condition). Then the configuration space is multiply connected. For example, consider two particles at $\mathbf{r}_1$ and $\mathbf{r}_2$. We can go to centre of mass and relative coordinates. The centre of mass motion of a system is always insensitive to the statistics of the particles constituting the system since statistics is a relative concept.

With the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we are given that $\psi(\mathbf{r})$ is defined only if $|\mathbf{r}| > a$. So the configuration space is the two-dimensional plane of $\mathbf{r}$ with a disc of radius a removed (Fig. 1). This space is multiply connected because the closed curve shown (which winds around the the disc once) cannot be continuously shrunk to a point.

Now imagine changing the coordinate $\mathbf{r} = (x,\ y)$, or rather, $z = x + i\ y$ to $z \exp(i2\pi)$. Since this is really the same point in the configuration space, the probability $|\psi|^2$ must be the same as before. But we could have a relative phase. That is,

$$\psi(ze^{i2\pi}, z^\star e^{-i2\pi}) = e^{i2\pi\nu}\psi(z, z^\star)$$

where ν is some real parameter.

Alternatively, for an anticlockwise exchange of the two particles i.e. $z \rightarrow ze^{i\pi}$, we would have

$$\psi(ze^{i\pi}, z^\star e^{-i\pi}) = e^{i\pi\nu}\psi(z, z^\star)$$

If $\nu = 0$ (or 1), the two particles are bosons (or fermions). But other values of ν cannot be ruled out on physical grounds. Actually, only the value of ν mod 2 matters in this argument.

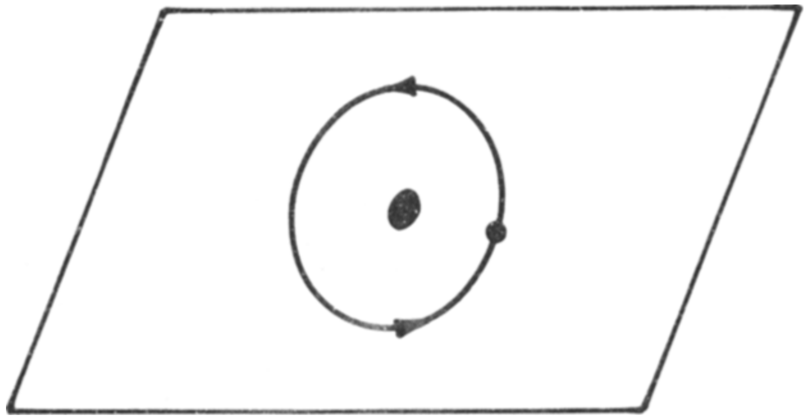

Figure 1. Configuration space for two identical particles with a hard core condition. A closed curve which cannot be shrunk to a point is shown.

The word 'anyons' has been coined to describe the possibility of particles having *any* statistics in two dimensions [1]. What kind of particles can have such statistics? Elementary particles like electrons or photons cannot show exotic statistics because there are deep and fundamental reasons why they are fermions or bosons. But there is no reason why a collective excitation of many electrons cannot behave like an anyon.

It has been known for quite some time that collective excitations can exhibit fractional *charge* (e.g. solitons in polyacetylene). Now imagine a situation in two dimensions where the magnetic flux threading a charged object is proportional to its charge (Fig. 2). Thus

$$\Phi = Kc\hbar Q,$$

where K is a constant independent of Q. Then if we take one such object around another, the wave function of the system picks up the Aharonov-Bohm phase

$$\exp(i\frac{Q}{c\hbar}\int \mathbf{A}\cdot d\mathbf{l}) = \exp(i\frac{Q}{c\hbar}\Phi) = \exp(\; i\; K\; Q^2\;)$$

If $Q = e$ (i.e. the objects are electrons with charge e), we must have

$$K = \frac{2\pi}{e^2}m,$$

where m is an odd integer in order that ν for such particles be equal to $m = 1$ mod 2. Now if there is another object with, say, $Q = e/m$, then $KQ^2 = 2\pi/m$. Then ν for these fractionally charged particles will be $1/m$ which is between 0 and 1 for $m = 3, 5, \cdots$. So fractionally charged particles in a system in which $\Phi \propto Q$ will necessarily show fractional statistics.

2. Fractional quantum Hall effect

Now we have to ask, what kind of a system can have

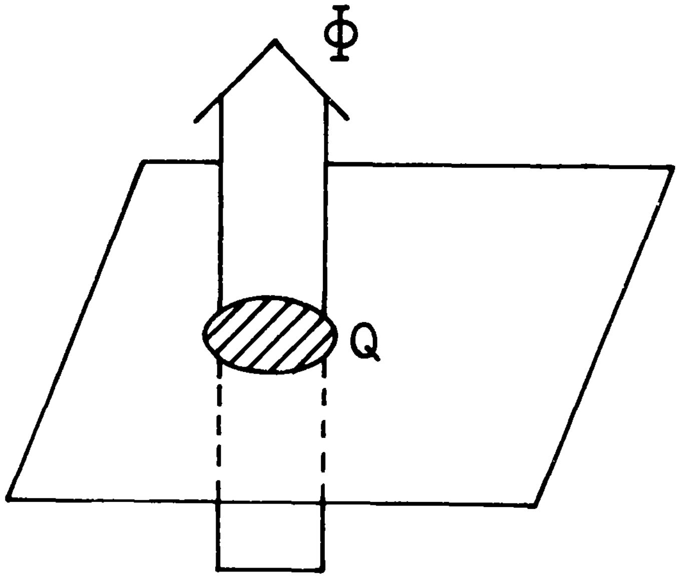

Figure 2. Magnetic flux Φ passing perpendicularly through a region carrying charge Q.

(i) fractionally charged particles, and
(ii) $\Phi \propto Q$?

The most well-known system of this kind is the one displaying the fractional quantum Hall effect (FQHE) [2,3]. This was discovered in 1982. Electrons in these systems are essentially confined to a two-dimensional layer and subject to a perpendicular magnetic field $B \sim 10$ Tesla. So the electrons occupy the various Landau states. These are shown schematically in Fig. 3. Suppose that the density of electrons per unit area $\rho < eB/2\pi$. Then all the electrons are spin-polarized and only states in the lowest Landau level are occupied at low temperatures. The filling fraction (i.e. the fraction of lowest Landau level states which are occupied) $= 2\pi\rho/eB$. Suppose that this equals 1/3. Then the Coulomb repulsion between the electrons leads to a naive picture of the occupied states as shown in Fig. 4. Actually the electrons do not sit in such an orderly (crystalline) manner, so this figure is only for purposes of illustration.

At low temperatures ($< 5^o K$), this system is found to have some unusual properties at specific values of the filling fraction like 2/5, 1/3, 2/7, 1/5, $\cdots$; for instance, the system is incompressible. Let us take the filling fraction to be 1/3 henceforth for simplicity. Then incompressibility means that even if the density (or filling fraction) is changed from 1/3 by a small amount (the easiest way to do this is to change B), the actual density continues to be 1/3 at all places, except for some isolated and localised regions where the deficit or excess density is made up. (This is schematically shown in Fig. 5). These regions have a size $\sim 100 A^o$. We will soon identify them with fractional charge and fractional statistics.

The excitations of the lowest energy (i.e. energy lower than that required to excite an electron to the next Landau level) consist of these local defects — either

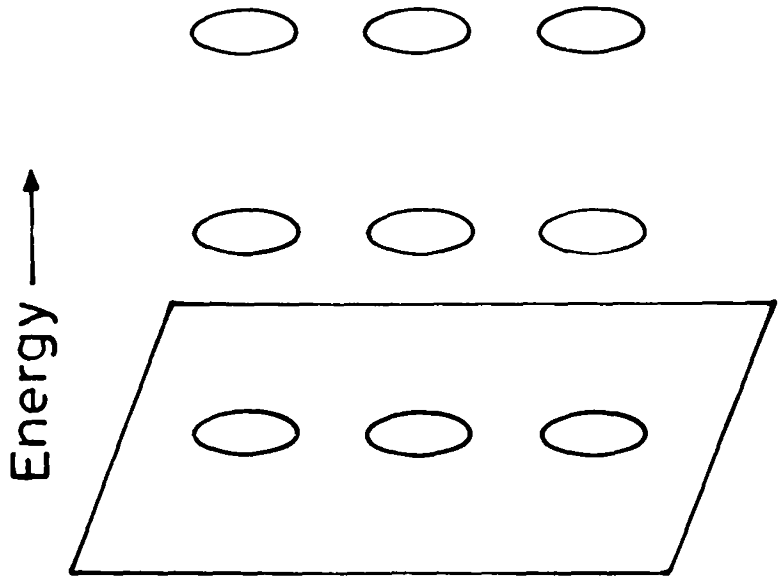

Figure 3. Landau levels in an uniform magnetic field. The energy spacing (shown vertically) is eB/μ and the 'Landau area' of each state is $2\pi/eB$.

a charge deficit (called a quasihole) or a charge surplus (called a quasielectron). We now argue that these carry fractional charge. The argument is easiest to visualise in one dimension as shown in Fig. 6. Note that the charge is measured with respect to the ground state. Similarly, in the FQHE with the filling fraction equal to 1/3 , the excitations above the ground state can be shown to have charges $\pm e/3$.

I should emphasize again that the electrons in the FQHE do *not* arrange themselves into a crystal as the figures above would suggest. Rather, they form an incompressible fluid with no long range order. Explaining that in detail would have taken us into some things called the Laughlin wave functions. My purpose here was only to illustrate in a simple way the idea of fractional charge because that is really the key concept here.

Having seen that there can be fractionally charged quasiparicles, let us now see why $\Phi \propto Q$. (Then fractional statistics would follow immediately).

There is a novel and very useful way of thinking about the FQHE [4]. Imagine doing a gauge transformation on the wave function of the electrons. The gauge transformation we want will associate a flux $\Phi = 3\Phi_0$ with each electron where Φ_0 = the elementary flux quantum = $2\pi\hbar c/e$. Also, we choose this flux to point opposite to the external magnetic filed B.

The mathematical way of doing this is the following. We start with $H\psi = E\psi$, where

$$H = -\frac{1}{2\mu}\sum_i (\nabla_i - \frac{ie}{c}\mathbf{A}(i))^2 + \text{ Coulomb potential},$$

where $\mathbf{A}$ is the vector potential such that $\nabla \times \mathbf{A} = \hat{z}B$. Then we do a gauge

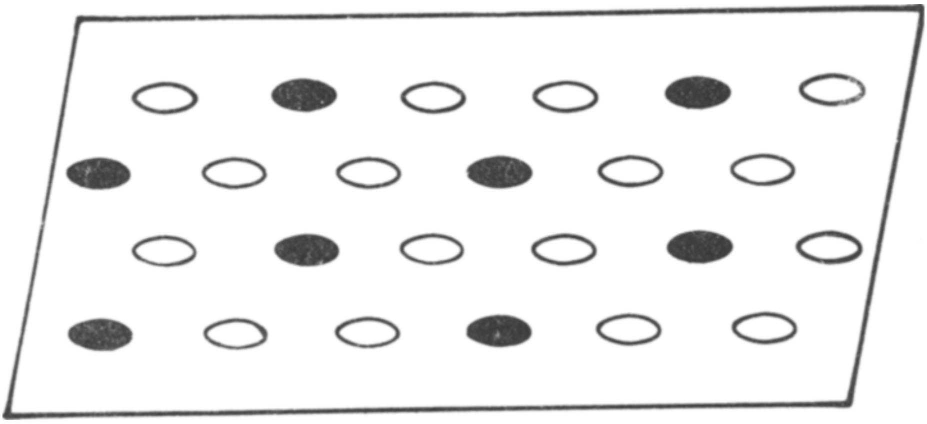

Figure 4. Schematic picture of occupied states in the lowest Landau level for filling fraction equal to 1/3.

transformation

$$\psi \rightarrow \psi'(\mathbf{r}) \;=\; \exp(\frac{ie}{c}\theta(\mathbf{r}))\psi(\mathbf{r}),$$

where $\mathbf{a} = \nabla\theta$ satisfies

$$\mathbf{b} = \nabla \times \mathbf{a} = \hat{z}3\Phi_0 \sum_i \delta^2(\mathbf{r} - \mathbf{r}_i).$$

This vector potential $\mathbf{a}$ and magnetic field $\mathbf{b}$ are not physical but are just mathematical constructs. We can call them fictitious fields. The new wavefunction ψ' satisfies $H'\psi' = E\psi'$, where $H' = \exp(ie\theta)H\exp(-ie\theta)$. Thus

$$H' = -\frac{1}{2\mu}\sum_i(\nabla_i - \frac{ie}{c}\mathbf{A}(i) - \frac{ie}{c}\mathbf{a}(i))^2 + \text{ Coulomb potential.}$$

Now, the 'particles' whose wave function is ψ' have the following two peculiar properties.

(i) They see no magnetic field on the average. This is because the external field $\mathbf{B}$ (which is uniform) and the fictitious field $\mathbf{b}$ (which is localised at the particles) cancel out on the average (Fig. 7).

(ii) The 'particles' are bosonic. This is because when one of these 'particles' is taken around another, the phase picked up due to the field $\mathbf{b}$ is

$$\exp(\frac{ie}{\hbar c}3\Phi_0) = \exp(i6\pi).$$

So an exchange would produce the phase $\exp(i3\pi) = -1$. But the original particles were fermions, so ψ is totally antisymmetric. Hence ψ' must be totally symmetric.

People, who use this formulation then go on to argue that these 'particles' can exhibit Bose condensation, there is an order papramater and so on. They develop a Landau-Ginzburg theory and explain all the observed properties of the FQHE

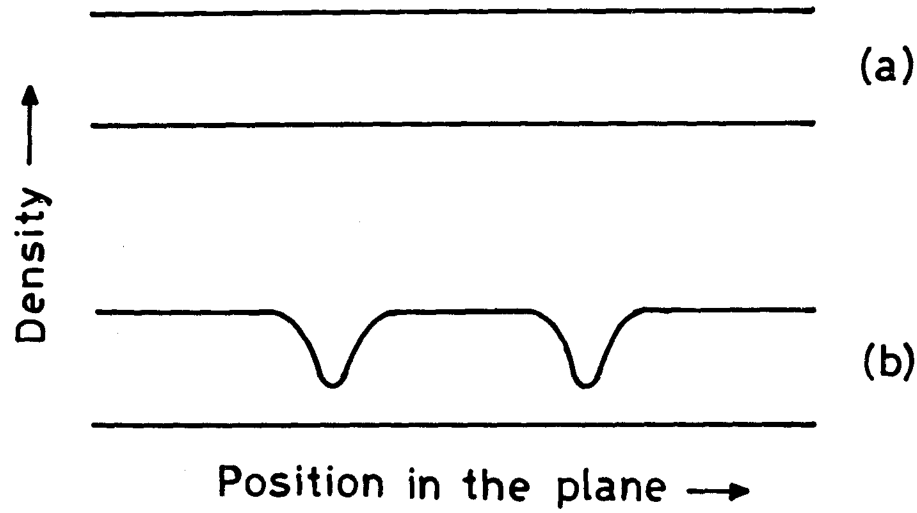

Figure 5. Sketches of density versus position in the plane for the ground state (top) and an excited state with two quasiholes (bottom).

using ideas from superconductivity. For our purpose, the main point is that we have a number of 'bosons' which move around in an average magnetic field which is zero. This is the ground state. Now suppose that there is a quasihole present somewhere. We saw earlier that this implies a local deficit of 1/3 of a particle or a local charge deficit of 1/3. In that region therefore, the external magnetic field is not completely cancelled. Hence, from a distance, that region appears to carry a magnetic flux $= 1/3(3\Phi_0) = \Phi_0$. The flux in this localised region (= sum of external and fictitious flux) is proportional to its charge, where both the flux and the charge are being measured with respect to the background (i.e. the ground state). Indeed, the constant of proportionality K can be easily found since $\Phi_0 = Kc\hbar(e/3)$ implies that

$$K = \frac{2\pi}{e^2} 3,$$

so that m equals the odd integer 3. Therefore, by our earlier argument, the quasiholes are anyons with $\nu = 1/3$.

3. Field theory description of anyons

We now come to another topic. Suppose that we wanted to describe just the quasiparticles (say, quasiholes) by some theory without making any reference to the background of electrons. The quasiholes have a charge $e^\star = -e/3$ which couples to the external magnetic field. Furthermore, they are described by the statistics parameter $\nu = 1/3$. A Hamiltonian with these features is

$$H = -\frac{1}{2\mu^\star} \sum_j (\nabla_j - \frac{ie^\star}{c}\mathbf{A}(j) - \frac{ie^\star}{c}\mathbf{a}(j))^2 + V_{Coulomb}.$$

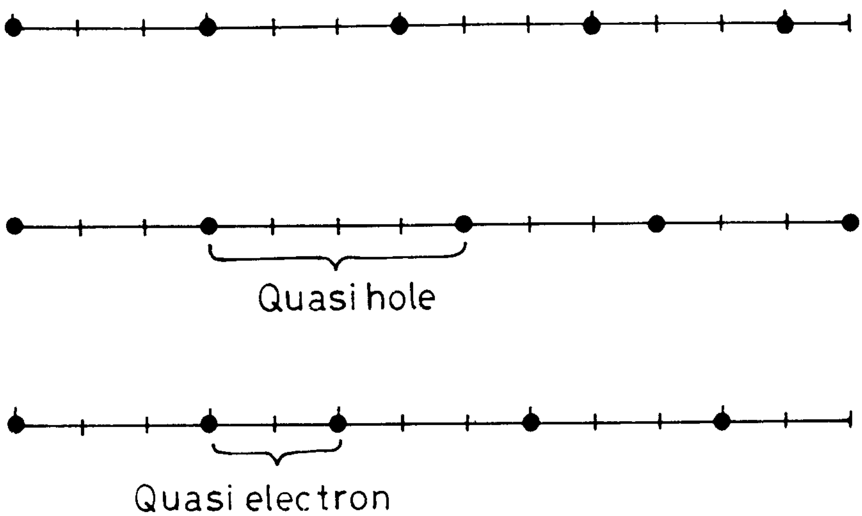

Figure 6. One-dimensional illustrations of occupied states in the ground state (top), a quasihole (middle) and a quasielectron state (bottom). The locations of the quasiparticles are also shown. These three states carry charges $Q = 0, -e/3$ and $e/3$ respectively.

(Note that the quasihole mass $\mu^\star$ is quite different from the electron mass μ). Here $\mathbf{a}$ is a different vector potential which produces a flux Φ_0 at the location of each quasihole. Thus

$$\mathbf{b} = \nabla \times \mathbf{a} = \hat{z}\Phi_0 \sum_j \delta^2(\mathbf{r} - \mathbf{r}_j).$$

A suitable $\mathbf{a}$ is

$$\mathbf{a} = \Phi_0 \hat{z} \times \sum_j \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|^2}.$$

The Hamiltonian above is assumed to act on totally symmetric (bosonic) wave functions. The fictitious field $\mathbf{a}$ then makes sure that the actual statistics of the particles is given by $\nu = 1/3$, because when two of them are exchanged, the phase picked up is

$$\exp(\frac{ie^\star}{c\hbar}\Phi_0/2) = \exp(i\pi/3).$$

Now, the $\mathbf{a}$ field is non-local. It would be more convenient if we could write down a *local* Hamiltonian or Lagrangian for the same system. Let us ignore the Coulomb potential and the external magnetic field for simplicity, and also use units in which $\hbar = c = 1$. Then a local Lagrangian which does the job is

$$L = \frac{\mu^\star}{2}\sum_j \dot{\mathbf{r}}_j^2 + e^\star \sum_j [-a_0(j) + \mathbf{a}(j)\cdot\dot{\mathbf{r}}_j] + \frac{e^{\star 2}}{4\pi\nu}\int d^2\mathbf{r}\,\epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda.$$

We have thus introduced three independent fields (a_0 , $\mathbf{a}$) which have their own dynamics due to the last term in L. This is called the Chern-Simons term.

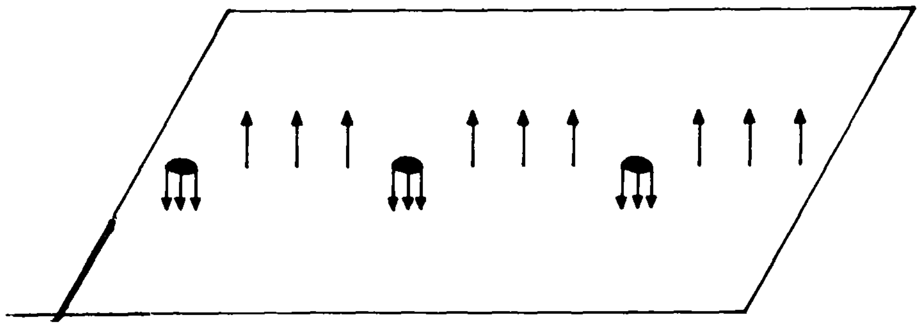

Figure 7. A picture of particles carrying a fictitious flux $3\Phi_0$ and moving in an uniform and opposing magnetic field.

The Chern-Simons term L_{C-S} has several interesting properties, two of which are the following.

(i) It is not gauge-invariant. But its contribution to the action

$$S_{C-S} = \int dt L_{C-S}$$

is gauge-invariant upto surface terms.

(ii) It is Lorentz invariant, even though the system we are studying is not Lorentz invariant.

It is easy to see that such a term contributes to the statistics of *any* particle to which the field a_μ (now called the Chern-Simons field) couples. Consider just the a_μ part of the Lagrangian density

$$\mathcal{L} = -J_\mu a^\mu + \frac{e^{\star 2}}{4\pi\nu}\epsilon_{\mu\nu\lambda}a^\mu\partial^\nu a^\lambda .$$

(The current J_μ has the property that a single-particle source gives $\int d^2\mathbf{r}J^0 = e^\star$). The equation of motion for a_μ is

$$J_\mu = \frac{e^{\star 2}}{2\pi\nu}\epsilon_{\mu\nu\lambda}\partial^\nu a^\lambda .$$

Thus

$$b = \frac{1}{2}\epsilon_{0ij}\partial^i a^j = \frac{2\pi\nu}{e^{\star 2}}J^0 .$$

This is again of the form magnetic flux $\Phi \propto$ charge Q if we integrate both sides of the equation over space. Now when one paricle is taken around another, we get the phase

$$\exp(ie^\star \int \mathbf{a}\cdot d\mathbf{l}) = \exp(\frac{i2\pi\nu}{e^\star}\int d^2\mathbf{r}J^0) = \exp(i2\pi\nu).$$

So if the particles producing the current J^{μ} were originally bosons (or fermions), the Chern-Simons field a_{μ} has the effect of 'shifting' the statistics by ν.

4. Final remarks

To be fair, I should say that an understanding of the FQHE in terms of anyons is not accepted by a large fraction of condensed-matter physicists. While it is not difficult to measure the charge of a quasiparticle, it is much more difficult to observe the relative statistics of two such objects. In fact, there has been *no* direct experimental evidence of fractional statistics so far. However, the concept of anyons has become very popular because of its novelty. Several physicists ranging from condensed-matter to nuclear physics have been studying various properties of anyons quite intensively for the last few years. I hope that this brief introduction to anyons will arouse the curiosity of some more people.

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Discussion

G. Rajasekaran : Just as anyons interpolete between bosons and fermions, can one envisage an interpoletion between parabosons and parafermions? Has anyone constructed such a theory?

D.Sen : Although no one has constructed such a theory to my knowledge, it should be possible in the following way. Take the Hamiltonian of parabosons in two dimensions. Assume t = c = 1 as well as the 'fictitious change' e=1. Attach a 'fictious flux' ϕ to each particle by a (singular) gauge transformation of the wave functions and the Hamiltanian. As ϕ is gradually changed from 0 to 2π, the phase of one exchange, which equals exp $(i\phi/c)$, changes from 1 to -1. Hence we should be going continuously frm parabosons to parafermions.

H.S. Mani : Can one have a second quantised version with the commutation relations of destruction and creation operator modified?

D. Sen : Yes. A second quantised form of arryons has been discussed by Frohlich and Marchetti in a Comm. Math. Phys. paper about 2-3 years ago. The commutation relations of anyons are more complicated than bosons or fermions and understanding them simply forces us to go from the complex plane to multiple Riemann sheets.

H.S. Mani : Is there any relation of these models to the one proposed by A.K. Mishra and G. Rajasekaran?

D. Sen : The anyon models discussed here have scalar or one-component wave functions. The models, discussed by Mishra and Rajasekaran, are equivalent to having multi-component wave functions and they are more related with parastatistics than with anyonic statistics.

R. Ramachandran : While anyons on a plane are describved by a single component wave function, there are multi-component wave functions to describe anyons on any compact surface. Even though they are of several components, they are not like the parastatistics systems. They resemble the type of new statistics as given by Rajasekaran and Mishra.

D. Sen : It is true that on a sphere or on a higher genus Riemann Surface, one has to use multi-component wave functions to describe anyons. (Using scalar wave functions restricts us to either fermions or bosons in contrast to the situation in a plane.) I did not know that such multi-component anyon wave functions were related to the new statistics described by Rajasekaran and Mishra. Thank you for the comment.

Probir Roy : Would you like to make some remarks on the Chern-Simon gauge theory?

D. Sen : It has been remarked by several authors in the last two years, like Fröhlich, Keiler, Wen and Zee, that the Landau-Ginzberg theory of the FQHE at exactly filling fractions equal to $\frac{1}{3}$, $\frac{1}{5}$ etc. is the pure Chern-Simon gauge theory. The CS theory correctly describes two well-known and experimentally observed properties of the FQHE on a finite surface e.g. a disc, viz. a) there are no low-energy excitations inside the system as there is a mass gap above the ground state, and b) there are massless and chiral excitations which run along the edges of the disc (called edge currents). Now, when the filling fraction is changed from the values $\frac{1}{3}$, etc., then some terms for the Chern-Simon field appear and then there are excitations inside the system which are anyons.