PRAMANA
— journal of
physics

Supplement to Vol. 41
December 1993
pp. 485–492

Bell's inequalities and quantum measurement theory

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Abstract. Bell's inequalities arising from the Einstein-Bell locality postulate or the Noncontextuality postulate provide valuable tests of the classical versus the quantum description independent of detailed dynamics. For 2^n degrees of freedom, these inequalities are violated by the quantum theory by a factor $2^{(n-1)/2}$ raising important questions for measurement theory.

1. Introduction

The desires for (i) Precision, (ii) Causality and (iii) Locality are three of the major motivations to look beyond present quantum theory. The standard quantum description splits the world into a 'system' described by a wave function and an 'apparatus' described by classical coordinates such as pointer positions. The 'Heisenberg split' between system and apparatus occurs at an ill-defined boundary; moreover quantum and classical descriptions can at best agree only approximately at the boundary. John Bell [1] thus argued that Quantum Mechanics is intrinsically ambiguous and approximate; a fundamental theory must attempt to do better. An obvious attempt to do better might be to extend the wave function description to the whole universe. But the time development implied by the linearity of the Schrödinger equation then leads us to consider 'grotesque' wave functions of the form

$$|\psi\rangle = \alpha|\psi, \text{here}\rangle + \beta|\psi, \text{there}\rangle$$

which is a superposition of macroscopically distinct configurations, even if we start from a wave function narrow on the macroscopic scale. Grotesque wave functions arise routinely in measurement type situations typified by the Schrödinger Cat Paradox [2]; yet, as Bohr stressed [3], the account of all evidence must be given in classical terms. In a picture in which the wave function is the complete description of the state, there must exist a mechanism of decoherence by which grotesque wave functions are reduced to wave functions or density matrices narrow on the macroscopic scale. There are now several such candidates [4] for an extended quantum mechanics.

The principle of causality on the other hand leads us to a picture in which the state is specified by supplementing the wave function by additional variables (hidden variables). Consider, e.g., photons passing successively through two linear polarizers with axes $\vec{a}$ and $\vec{b}$. A photon which has passed the first polarizer has its wave function specified (by the polarization axis $\vec{a}$); yet quantum mechanics does not tell us whether the photon will pass through the second polarizer; it only tells us that the probability of passage is $|\vec{a}\cdot\vec{b}|^2$. If we postulate the principle of causality:

Different outcomes have different causes,

then we must conclude that the photons which passed the second polarizer must have had some properties different from properties of those photons which did not. Since the wave function does not reveal the difference, these properties must be additional variables.

Einstein, Podolsky and Rosen [5] came to the same conclusion that the wave function must be an incomplete description by using an entirely different principle, viz. "Einstein locality". For the case of two systems S_1 and S_2 which have interacted in the past but are now spatially separated, the principle states [6] "the real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former". On formulating Einstein locality in precise mathematical terms, Bell [7] made the shocking discovery that the locality postulate contradicted the statistical predictions of quantum theory. It has since been verified that two photon correlation experiments [8] violate Einstein-Bell locality predictions (Bell's inequalities), and respect quantum theory predictions.

If we envisage then the possibility of hidden variable theories, these variables should not be constrained to obey locality. An example of a hidden variable theory violating locality is the De-Broglie-Bohm theory [9]. In such a theory of the universe, a grotesque wave function is no problem of principle, because the precise positions of objects are given in classical terms by the additional (hidden) variables. No mechanism of decoherence is needed.

If grotesque wave functions exist, they might be signalled by characteristic macroscopic interference phenomena such as those considered by Leggett et al [10] for SQUIDS. Here we shall use the violation of Bell inequalities as a measure of departure from classical behaviour. We discuss recent results [11,12] which show that for a system with 2^n degrees of freedom quantum theory (with grotesque states) leads to a violation of classical behaviour growing exponentially with n. Possible implications for quantum measurement theory will be discussed.

2. Bell's inequalities from Einstein locality

Consider two particles flying apart from an apparatus S into two instruments which measure physical quantities A and B with instrument setting parameters a and b respectively. We assume that the measurements occur in spacelike separated regions, and that, by the very definition of A, B, $|A| \leq 1$ and $|B| \leq 1$ (e.g. $A = +1$ for transmission through one channel and -1 for transmission through the other channel of a two-channel polarizer). Let S produce the two particles in a state characterized collectively by variables (including hidden variables) λ with probability distribution $\rho(\lambda)$. Then Einstein-Bell locality means that in this state the expectation value $\bar{A}(\lambda, a)$ of A can depend on λ and the instrument settings a, but not on the far away instrument settings b. With a similar argument for B, we deduce that for N coincidence detections

$$P(a,b) \equiv \langle AB \rangle \equiv \frac{1}{N}\sum_{i=1}^{N} A_i B_i = \int d\lambda \rho(\lambda)\bar{A}(\lambda,a)\bar{B}(\lambda,b), \tag{1}$$

where

$$\rho(\lambda) \geq 0, \quad \int d\lambda \rho(\lambda) = 1, \quad |\bar{A}(\lambda,a)| \leq 1, \quad |\bar{B}(\lambda,b)| \leq 1. \tag{2}$$

The elementary inequality

$$|x(y-y')| + |x'(y+y')| \leq 2\ \max(|x|,|x'|)\max(|y|,|y'|)$$

then leads to the Bell-CHSH inequality

$$|P(a,b) - P(a,b')| + |P(a',b) + P(a',b')| \leq 2. \tag{3}$$

In the Bohm-Aharonov example of a pair of spin half particles in the singlet state ψ, the quantum mechanical correlation function is

$$P_{QM}(a,b) = \langle \psi | \vec{\sigma}_1 \cdot \vec{a} \vec{\sigma}_2 \cdot \vec{b} | \psi \rangle = -\vec{a} \cdot \vec{b}, \tag{4}$$

which makes the left-hand side of Eq. (3) equal $2\sqrt{2}$ for the choice $\vec{a} = (0,1,0)$, $\vec{b} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $\vec{a}\,' = (1,0,0)$, $\vec{b}\,' = (1/\sqrt{2}, -1/\sqrt{2}, 0)$. This proves Bell's theorem: the statistical predictions of quantum theory violate Einstein-Bell locality.

A similar violation of locality is predicted for the two-photon case by quantum theory and has been confirmed experimentally by beautiful experiments [8]. Even if Bell's inequalities are violated for few particle systems, they are useful because they yield a quantitative measure of the departure from classical behaviour. In quantum measurement theory one expects to regain classical description for macroscopic systems. Bell's inequalities for multiparticle systems can thus be a valuable tool to test the validity of the classical description for macroscopic systems.

Consider a system of particles $1, 2, \cdots, n$ flying apart from an apparatus S producing them. Let the physical quantities $A_1, A_2, \cdots, A_n$ be measured for them by instruments in spacelike separated regions, settings of the instrument measuring the j^{th} particle being denoted collectively by a_j. We assume that by their very definition $|A_i| \leq 1$. According to Einstein-Bell locality, the n-particle correlation function has the representation,

$$P(a_1, \cdots, a_n) = \langle \prod_{i=1}^{n} A_i \rangle = \int d\lambda \rho(\lambda) \prod_{i=1}^{n} \bar{A}_i(\lambda, a_i), \tag{5}$$

with

$$\rho(\lambda) \geq 0, \quad \int d\lambda \rho(\lambda) = 1, \quad |\bar{A}_i(\lambda, a_i)| \leq 1. \tag{6}$$

Here locality has been used to require that the probability distribution $\rho(\lambda)$ of the hidden variables of the initial state produced at S is independent of the a_i, and each $A_i(\lambda, a_i)$ is independent of the far away apparatus parametes a_j (with $j \neq i$).

In a quantum theory obeying microcausality the physical quantities A_i are represented by eigenvalues of mutually commuting self-adjoint operators $A_i(a_i)$; in a quantum state ψ the correlation function is

$$P_\psi(a_1,\cdots,a_n) = \langle\psi|\prod_{i=1}^{n} A_i(a_i)|\psi\rangle. \tag{7}$$

Mermin [11] used a state ψ of n spin 1/2 particles (which was exploited earlier by Greenberger, Horne and Zeilinger [13] to present a proof of Bell's theorem without inequalities). He derived elegant n-particle Bell inequalities which are violated by quantum correlation functions by a factor $2^{(n-1)/2}$ for odd n, and $2^{n/2-1}$ for even n. Roy and Singh [11] used states more general than the GHZ state, and derived new Bell inequalities which are violated by quantum correlations by a factor $2^{(n-1)/2}$ for both odd and even n.

We now derive inequalities on linear combinations of the correlation functions $P(a_1,\cdots,a_n)$ using the methods of MRS [11]. Consider

$$\begin{aligned} &F^{(n)} = \textstyle\prod_{i=1}^{n}\left[A_i(a_i) + i\eta_i A_i(a_i')\right], \quad A^{(n)} = (F^{(n)} + F^{(n)\dagger})/2, \\ &B^{(n)} = (F^{(n)} - F^{(n)\dagger})/(2i), \end{aligned} \tag{8}$$

where $\eta_i = \pm 1$. Then, $A_\psi^{(n)} = \langle\psi|A^{(n)}|\psi\rangle$, $B_\psi^{(n)} = \langle\psi|B^{(n)}|\psi\rangle$ involve only linear combinations of quantum correlation functions in the state ψ. Their hidden variable analogues are

$$A_{HV}^{(n)} = \int d\lambda\rho(\lambda)A^{(n)}(\lambda), \quad B_{HV}^{(n)} = \int d\lambda\rho(\lambda)B^{(n)}(\lambda), \tag{9}$$

where $A^{(n)}(\lambda) = Re F^{(n)}(\lambda)$, $B^{(n)}(\lambda) = Im\ F^{(n)}(\lambda)$, and

$$F^{(n)}(\lambda) = \prod_{i=1}^{n}\left[\bar{A}_i(\lambda,a_i) + i\eta_i\bar{A}_i(\lambda,a_i')\right]. \tag{10}$$

Since the $A^{(n)}(\lambda)$ and $B^{(n)}(\lambda)$ are linear functions of each of the arguments $\bar{A}_i(\lambda,a_i)$ and $\bar{A}_i(\lambda,a_i')$, their maxima and minima - on varying the arguments between -1 and $+1$ - must be reached on the boundary, i.e., for each argument must be ± 1. It follows that

$$|A^{(n)}(\lambda)| \le p_n, \quad |B^{(n)}(\lambda)| \le p_n \tag{11}$$

and hence

$$|A_{HV}^{(n)}| \le p_n, \quad |B_{HV}^{(n)}| \le p_n, \tag{12}$$

where

$$p_n = 2^{(n-1)/2} \text{ for } n \text{ odd}; \; p_n = 2^{n/2} \text{ for } n \text{ even}. \tag{13}$$

We now show that these locality inequalities can be violated by quantum correlations. Consider n spin 1/2 particles and $A_i(a_i) = \vec{\sigma}_i\cdot\vec{a}_i$. Choose all $\eta_i = +1$, all $\vec{a}_i = \hat{x}$, all $\vec{a}_i{}' = \hat{y}$, and for $|\psi\rangle$ consider the choices

$$|\phi_\pm^{(n)}\rangle = (|++\cdots+\rangle \pm i|--\cdots-\rangle)/\sqrt{2}, \quad |\chi_\pm^{(n)}\rangle = (|+\cdots+\rangle \pm |-\cdots-\rangle)/\sqrt{2}, \tag{14}$$

where the sign $\pm$ within the kets on the right-hand side at the j^{th} place denotes eigenvalues ± 1 of σ_Z for the j^{th} particle. Then,

$$(A^{(n)} \mp 2^{n-1})|\chi_{\pm}^{(n)}\rangle = 0, \quad (B^{(n)} \mp 2^{n-1})|\phi_{\pm}^{(n)}\rangle = 0; \tag{15}$$

hence the quantum mechanical expectation values $A_{\psi}^{(n)}$ and $B_{\psi}^{(n)}$ equal $\pm 2^{n-1}$ for $\psi = \chi_{\pm}^{(n)}$ and $\phi_{\pm}^{(n)}$ respectively, violating the locality bounds (12) by a factor $2^{(N-1)/2}$, with $N = n$ for n odd, and $N = n-1$ for n even.

For even n, Roy and Singh [11] obtained an improved inequality. Using (11) we have

$$\begin{aligned} I_n \equiv & \left| \int d\lambda \rho(\lambda) B^{(n-1)}(\lambda)(\bar{A}_n(\lambda, a_n) + \bar{A}_n(\lambda, a_n')) \right| \\ & + \left| \int d\lambda \rho(\lambda) A^{(n-1)}(\lambda)(\bar{A}_n(\lambda, a_n) - \bar{A}_n(\lambda, a_n')) \right| \\ & \leq 2^{n/2}, \text{ if } n \text{ is even.} \end{aligned} \tag{16}$$

The quantum mechanical value corresponding to the left-hand side of (16) is

$$\begin{aligned} I_n(\psi) = & |\langle\psi|B^{(n-1)}(A_n(a_n) + A_n(a_n'))|\psi\rangle| \\ & + |\langle\psi|A^{(n-1)}(A_n(a_n) - A_n(a_n'))|\psi\rangle|. \end{aligned} \tag{17}$$

Choosing as before $\eta_i = 1$, $\vec{a}_i = \hat{x}$, $\vec{a}_i{}' = \hat{y}$ for $i \leq n-1$, but keeping $\vec{a}_n$ arbitrary, and defining

$$\begin{aligned} (|\psi\rangle, |\psi'\rangle) \equiv & \tfrac{1}{\sqrt{2}}((|\phi_+^{(n-1)}\rangle, |\chi_+^{(n-1)}\rangle)|-\rangle \\ & -(|\phi_-^{(n-1)}\rangle, |\chi_-^{(n-1)}\rangle)|+\rangle), \end{aligned} \tag{18}$$

we have,

$$\langle\psi|(A^{(n-1)}, B^{(n-1)})\vec{\sigma}_n \cdot \vec{a}_n|\psi\rangle = -2^{n-2}((a_n)_y, (a_n)_z). \tag{19}$$

Further, choosing $\vec{a}_n = (0, 1/\sqrt{2}, 1/\sqrt{2})$, and $\vec{a}_n' = (0, \mp 1/\sqrt{2}, \pm 1/\sqrt{2})$ for the states $|\psi\rangle$, $|\psi'\rangle$, we obtain,

$$I_n(\psi) = I_n(\psi') = 2^{(2n-1)/2}, \tag{20}$$

which violate the locality inequality (16) by a factor $2^{(n-1)/2}$ for even n.

3. Bell's inequalities from noncontextuality

In the quantum theory, the expectation value of an observable A is unaffected by the previous (or simultaneous) measurement of any set of mutually commuting observables commuting with A. This 'statistical noncontextuality' provides the inspiration for the 'noncontextuality' hypothesis in hidden variable theories or deeper level theories. In a deterministic noncontextual hidden variable theory, the value of an observable A is independent of which particular set of mutually commuting observables commuting with A are measured together with A. The Gleason and Kochen-Specker theorems [14] prove the impossibility of such a hidden variable theory for quantum mechanics. A particularly interesting case of 'noncontextuality' arises when observables A and B commute due to spacelike separation, and the noncontextuality hypothesis becomes the Einstein locality hypothesis which we discussed before. In that case Bell's theorem proves a stronger result, viz. the impossibility, not only of deterministic but also of stochastic local hidden variable theory for quantum mechanics. Recently [12] the Gleason-Kochen-Specker theorem too has been extended to stochastic noncontextual hidden variable theories. Further, for a single particle of spin S, with $2S+1=2^n$, n being an integer, it has been shown that the quantum violation of noncontextuality grows exponentially with n. These violations are novel illustrations of the quantum theory conflicting with the classical idea of noncontextual realism. Evidently, noncontextuality is more stringent than locality because it applies to a greater variety of physical situations, e.g. for a *single* high spin particle there are no locality inequalities.

Stochastic noncontextual hidden variable theories

Let $A_1(a_1), \cdots, A_n(a_n)$ be dynamical variables corresponding to different degrees of freedom of a given physical system, the settings of the apparatus measuring $A_j(a_j)$ being denoted by a_j. In the quantum theory the dynamical variables are represented by observables $A_i(a_i)$, with

$$[A_i(a_i), A_j(a_j)] = 0. \tag{21}$$

In a hidden variable theory, the state of the system may be characterized by variables λ (which may include the quantum state vector as well), with $\rho(\lambda)$ being their probability distribution. For given λ, the dynamical variables have expectation values $\bar{A}_1(\lambda, a_1)$, $\bar{A}_2(\lambda, a_2)$, $\cdots$, $\bar{A}_n(\lambda, a_n)$, noncontextuality implying that $\bar{A}_i$ depends only on λ and a_i, but not on a_j with $j \neq i$. We assume that by the very definition of the A_i, $|A_i| \leq 1$ (e.g. $A_i = +1$ for transmission through a polarizer and -1 for nontransmission), and hence that

$$|\bar{A}_i(\lambda, a_i)| \leq 1, \ \forall i. \tag{22}$$

Further, in complete analogy to Bell's argument for local stochastic theories [3] we define noncontextual stochastic theories to be those in which $\overline{A_1 A_2}(\lambda, a_1, a_2) = \bar{A}_1(\lambda, a_1)\bar{A}_2(\lambda, a_2)$. To motivate this, suppose that $\bar{A}_i(\lambda, a_i)$ is the average over hidden variables λ_i of the apparatus which measures A_i to be $A_i(\lambda, \lambda_i, a_i)$. Noncontextuality requires that $A_1(\lambda, \lambda_1, a_1)$ and the probability distribution $\rho_1(\lambda_1)$ of λ_1 must be independent of which commuting variables are measured together with A_1 and in particular of λ_2, a_2. Hence

$$\overline{A_1 A_2}(\lambda, a_1, a_2) = \int d\lambda_1 d\lambda_2 \rho_1(\lambda_1)\rho_2(\lambda_2)$$
$$\times A_1(\lambda, \lambda_1, a_1) A_2(\lambda, \lambda_2, a_2) = \bar{A}_1(\lambda, a_1)\bar{A}_2(\lambda, a_2).$$

The correlation function for the n-variables (which could even refer to commuting observables for a single particle) thus has the representation (5), (6) for the noncontextual stochastic hidden variable theory and the representation (7) for quantum mechanics. The Bell inequalities (12) and (16) therefore follow exactly as in the last section, and may be compared with quantum mechanical results in specific examples.

Power law violation of classical behaviour for a particle of high spin

A striking consequence of the above formulation of stochastic noncontextual theories (entirely outside the scope of Bell's locality theorem) is the following result.

For a single particle of spin $S = (2^n - 1)/2$, where $n = 1,2,3,\cdots$, the quantum theory violates noncontextual realism by a factor $(S+1/2)^{1/2}$ if n is odd, and $(S+1/2)^{1/2}/\sqrt{2}$ if n is even.

A particle with spin S may be described quantum mechanically by means of a $(2S+1)$ component wave function ψ in a suitable orthonormal basis $|\alpha\rangle$. When $2S+1 = 2^n$, we may choose the labels α to be n-tuples: $\alpha \equiv m_1 m_2 \cdots m_n$, where $m_i = \pm 1$ (or simply $m_i = \pm$). Thus

$$|\psi\rangle = \sum_\alpha \psi_\alpha |\alpha\rangle, \quad \alpha \equiv m_1 m_2 \cdots m_n, m_i = \pm. \tag{23}$$

Consider the hermitian operators $A_i(a_i)$ with matrix elements

$$\langle \alpha' | A_i(a_i) | \alpha \rangle = (\vec{\sigma} \cdot \vec{a}_i)_{m_i' m_i} \prod_{j \neq i} \delta_{m_j' m_j}, \quad i = 1, \cdots, n, \tag{24}$$

where $\alpha' \equiv m_1' \cdots m_n'$, and the $\vec{\sigma}$ are Pauli matrices, i.e.

$$\vec{\sigma} \cdot \vec{a} = \begin{pmatrix} a_z & a_x - i a_y \\ a_x + i a_y & -a_z \end{pmatrix}.$$

Then $A_i^2(a_i)$ equals the unit matrix and hence each $A_i(a_i)$ has eigenvalues ± 1. Further, for $i \neq j$,

$$\langle \alpha' | A_i(a_i) A_j(a_j) | \alpha \rangle = (\vec{\sigma} \cdot \vec{a}_i)_{m_i' m_i} (\vec{\sigma} \cdot \vec{a}_j)_{m_j' m_j} \prod_{k \neq i,j} \delta_{m_k' m_k},$$

which shows that the $A_1(a_1), \cdots, A_n(a_n)$ are mutually commuting operators. Hence we obtain the representation (5) for their correlation function in stochastic noncontextual theories. With the notation (23), choosing for $|\psi\rangle$ the wave functions (14), we again obtain Eq. (15) which violate the bounds (12) (this time derived from noncontextuality) by a factor growing like $\sqrt{S}$ for large S.

4. Consequences for measurement theory

We found a quantum violation of Einstein-locality in a system of n spin $1/2$ particles and a quantum violation of noncontextuality for a single particle of spin $S = (2^n - 1)/2$. The two systems have the same number of degrees of freedom 2^n; the violations in both cases grow exponentially in n or as the square root of the number of degrees of freedom. Thus, it might be generally true that for a system with large number of degrees of freedom, quantum theory implies violation of classical behaviour growing like square root of the number of degrees of freedom. Note however that the quantum states such as (14) and (18) leading to these large violations are grotesque states. If such large violations are not seen in nature, we must look for an extended quantum theory which suppresses grotesque states. The particular extension due to GRW [4] postulates a spontaneous localization mechanism which makes the spatial wave function narrow on the macroscopic scale. This might suppress violations of Bell's inequalities following from locality; but it contains no mechanism to suppress the grotesque wave function for a single particle of high spin. A more general mechanism of decoherence will be needed for that.

Acknowledgements

I warmly acknowledge a long collaboration with Virendra Singh during which all the ideas reviewed here were developed.

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