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A semi-empirical formula for the energy spacings in Λ-hypernuclear excitation spectra

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Abstract. A semi-empirical formula is presented which gives a semi-quantitative account of the excitation energies in Λ-hypernuclear excitation spectra. Many elaborate studies made were numerical, but our studies yield an analytical expression. Different three-dimensional harmonic oscillators for the Λ- and the neutron-hole reproduce the observed energy spacings fairly well. However, it would be misleading that the oscillators give the wave functions of the corresponding states also. In the light of harmonic excitations, we briefly discuss some phenomenological and microscopic Λ-hypernuclear calculations.

Keywords. Semi-empirical; Λ-hypernuclear excitation energy; hyperon-nucleon interactions.

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1. Introduction

Λ-hypernuclei has been studied extensively over the last three decades. In the last decade, the strangeness exchange reaction (K^-, π^-) on nuclei has played a central role in hypernuclear spectroscopy. These reactions are characterized by small momentum transfer and are effective in producing substitutional states in which Λ has the same orbital as the neutron it replaces. They populate preferentially states of low angular momentum. In (K^-, π^-) spectra of medium and heavy nuclei, the states corresponding to deep-lying hyperon orbitals are generally not observed.

Recently, the associated production reaction (π^+, K^+) has been found to be effective in populating the deeply bound Λ-single particle levels in heavy hypernuclei. For nuclei over a large mass number range, Λ-single particle states have been obtained by this method [1–3].

Some of the earlier Λ-hypernuclear spectra were analysed by Bouyssy [4] who gave an apparently satisfactory account of the then available data by assuming pure particle-hole states in a single particle Λ-nucleus Woods-Saxon (WS) potential with an empirical term giving the nucleon, rather the neutron-hole spin-orbit splittings. A residual interaction was assumed which enabled removal of degeneracy of the different J-values resulting from the configurations. This made it possible to assign the J to the given state. At about this time, an alternative approach was adopted [5–7] to describe the states observed in the reactions as a coherent sum of particle-hole states designated as analog and supersymmetric. At that time, it was felt that the angular distributions would decide between the pure particle-hole states of Bouyssy [4] and the mixed states considered by Dalitz and Gal [5] and others.

With the analysis of the angular distribution, the matter seems to have been decided definitely in favour of the mixed states [6, 7], but that is not our concern here.

Without the slightest prejudice to the accepted nature of these states, we find that the Λ-hypernuclear excitation energy over the whole range of mass numbers can be described fairly well as the combined excitation of a three-dimensional harmonic oscillator describing the Λ and another, the neutron-hole (N_h) along with a suitable spin-orbit potential for the N_h. We do not suggest even obliquely that the oscillators give the corresponding wave functions, but the fact remains that they certainly give a semi-quantitative account of the energy spacings. The frequency of the Λ-oscillator is described by two adjustable parameters and that of the N_h by a single parameter only. Another adjustable parameter is for the spin-orbit splitting. The oscillator has been broadly mentioned in the context of the spectra, but no one seems to have performed such calculations earlier. However, besides Bouyssy [4], Dalitz and Gal [5], many numerical calculations have been carried out which gave a good account of the binding energies, cross-sections, angular distributions etc. Λ-single-particle energies have also been successfully analysed in the Skyrme Hartree-Fock approach in terms of Λ-nucleus potentials [8] and taking three-body ΛNN force [9] into account. A comprehensive review of Λ-single-particle states produced in (K^-, π^-) and (π^+, K^+) experiments has been given by Bando *et al* [10]. Many elaborate calculations discussed are numerical, whereas the present study is analytical, although more approximate.

As most of the data used here is obtained from the diagrams given in the literature and also as experimental errors in all the binding energies considered here are not available, we have assigned a plausible value to experimental errors to construct what we call the simulated chi-square (χ^2). This has been done simply to provide a criterion for adjusting the parameters and judging the quality of the fit.

Wave function notwithstanding, Bouyssy [4] was able to give a satisfactory account of the energy spacings by performing numerical calculations with his potential. Our work is broadly in the spirit of Bouyssy and the other workers, except that we confine ourselves to predicting the energy spacings. Use of the oscillators gives us an analytical expression, though certainly somewhat crude, for the energy spacings. It is clear that the fair amount of success achieved here should not be taken as any indication of the correctness of Bouyssy's philosophy of pure particle-hole states. Our objective is very limited, i.e., to give a semi-empirical formula for the energy spacings of the observed spectra. The analytical expressions, though approximate, enables one to see some dependences.

The quantum numbers of the outermost neutron orbit, required for the calculation of the energy spacings, as given by (13) in §3, are taken to be those given by the standard shell model sequence of the single-particle states. The successive excited states are obtained by varying the quantum numbers of the two oscillators and these are listed in table 1 to enable a quick check to be made on the calculations.

2. General hypernuclear excitations

The picture required for our limited purpose has already been mentioned in the introduction. This is the basis of the calculations leading to the semi-empirical formula for the energy spacings. We do not take this picture literally. It is useful to us in so far as it gives a fairly good account of the observed spectral energy spacings.

In general, we assume that the hypernuclear spectra arise from Λ- and N_h-excitations in their respective harmonic oscillator potentials. Thus, the excitation energy of the hypernucleus can be considered to be the sum of Λ- and the N_h- excitations. The

Λ-excitation is measured from the ground state of Λ-particle, while the N_h-excitation from the topmost occupied neutron level where a hole corresponds to an unexcited core. The sum of these two quantities is the total excitation energy of the system.

The Λ-excitation energy E_Λ^* is given as

$$E_\Lambda^* = E_{nl} - E_{10}, \tag{1}$$

where E_{nl} is the energy eigenvalue of the Λ-3-dimensional isotropic harmonic oscillator corresponding to (nl)-state. It is given as

$$E_{nl} = (2n_\Lambda + l_\Lambda - \tfrac{1}{2})\hbar\omega_\Lambda, \tag{2}$$

where $n_\Lambda = 1, 2, 3\ldots$ and $l_\Lambda = 0, 1, 2\ldots$; E_{10} is the oscillator ground state energy corresponding to $n_\Lambda = 1$ and $l_\Lambda = 0$. The Λ-oscillator frequency [11] is taken to be of the form

$$\hbar\omega_\Lambda = C_{\Lambda 1} A_c^{-1/3} - C_{\Lambda 2} A_c^{-2/3}, \tag{3}$$

where $C_{\Lambda 1}$ and $C_{\Lambda 2}$ are parameters to be fixed from analysis of the data.

As the spin-orbit splitting for Λ is known to be extremely small [4] it is not taken into consideration. Consequently, the Λ-energy is labelled by (nl) only. However, spin-orbit splitting has to be taken into account for N_h, the energy of which is, therefore, labelled by (nlj).

Employing a 3-dimensional isotropic harmonic oscillator for the N_h as well, its excitation energy can be written as

$$E_{N_h}^* = E_{n_1 l_1 j_1} - E_{n_2 l_2 j_2}, \tag{4}$$

where $(n_1 l_1 j_1)$ represents the topmost oscillator orbital occupied by neutrons and $(n_2 l_2 j_2)$ is the oscillator orbital which is actually occupied by the N_h. We have

$$E_{nlj} = (2n + l - \tfrac{1}{2})\hbar\omega_{N_h} + \tfrac{1}{2}[j(j+1) - l(l+1) - \tfrac{3}{4}] \times \int \psi_{nl}^*(r)\, V_{N_h}^{ls}(r)\psi_{nl}(r)\mathrm{d}\tau, \tag{5}$$

where $\psi_{nl}(r)$ is the harmonic oscillator wave function and the N_h spin-orbit potential is chosen, for convenience, to be of the form

$$V_{N_h}^{ls}(r) = \bar{v}_{N_h}^{ls} g(r), \tag{6}$$

where the radial dependence of the N_h spin-orbit potential is taken to be of the form

$$g(r) = \frac{r \cdot r_0^3}{\sqrt{2}R^4} \exp[-(r^2/2R^2)], \tag{7}$$

where $R = r_0 A_c^{1/3}$, r_0 being a parameter fixed at a fairly reasonable value of 1·0 fm. Small changes in r_0 necessitate changes in $\bar{v}_{N_h}^{ls}$ without spoiling the fit. The surface peaking of $g(r)$ is in accord with well established ideas about the spin-orbit potential. For harmonic oscillator wave function and $V_{N_h}^{ls}(r)$ given above, eq. (5) reduces to (for

Table 1. Predicted and observed energy spacings of the excited states. The observed energies are given in the parenthesis below the predicted value. Also given are the quantum numbers of Λ and N_h represented as (nl, nlj) for a given state, where nl are the quantum numbers of Λ and nlj those of N_h. Values marked by a star are those which do not involve any N_h-transition in our picture of the oscillators. In $^{40}_{\Lambda}$Ca, there are 5 more predicted excited states out of which the 15th predicted excited state has energy 30·84 MeV, whereas the observed value is 31·50 MeV. Only one energy level has been observed in $^{209}_{\Lambda}$Bi which according to our calculations seems to be the 38th excited state.

Hyper-nuclei	Energy spacings of the excited states (MeV) numbered as 1, 2, 3, ...									
	1	2	3	4	5	6	7	8	9	10
	10·85*	19·18	21·71*	30·04						
$^{6}_{\Lambda}$Li	(10·00)	(19·00)	(−)	(28·50)						
	$(11, 11\frac{3}{2})$	$(10, 10\frac{1}{2})$	$(12, 11\frac{3}{2})$	$(11, 10\frac{1}{2})$						
	10·87	19·59	21·73*	30·46						
$^{7}_{\Lambda}$Li	(10·50)	(21·50)	(−)	(30·50)						
	$(11, 11\frac{3}{2})$	$(10, 10\frac{1}{2})$	$(12, 11\frac{3}{2})$	$(11, 10\frac{1}{2})$						
	10·74*									
$^{9}_{\Lambda}$Be	(13·00)									
	$(11, 11\frac{3}{2})$									
	10·44*	19·37								
$^{12}_{\Lambda}$C	(11·50)	(19·00)								
	$(11, 11\frac{3}{2})$	$(10, 10\frac{1}{2})$								
	10·33*									
$^{13}_{\Lambda}$C	(10·80)									
	$(11, 11\frac{5}{2})$									
	6·51	10·03*	18·54							
$^{16}_{\Lambda}$O	(6·50)	(11·00)	(17·00)							
	$(10, 11\frac{5}{2})$	$(11, 11\frac{1}{2})$	$(11, 11\frac{5}{2})$							
	8·14*	15·82	18·27*							
$^{27}_{\Lambda}$Al	(8·00)	(−)	(17·00)							
	$(11, 12\frac{5}{2})$	$(10, 11\frac{5}{2})$	$(12, 12\frac{5}{2})$							
	9·07*	15·72	18·14*							

$^{28}_{\Lambda}$Si	(8·80)	(–)	(18·00)							
	$(11, 12\frac{5}{2})$	$(10, 11\frac{3}{2})$	$(12, 12\frac{5}{2})$							
$^{32}_{\Lambda}$S	8·83*	11·17	17·65*	18·21	20·54	24·00	26·48*	27·04	29·37	
	(–)	(11·00)	(–)	(–)	(20·00)	(–)	(–)	(–)	(30·50)	
	$(11, 20\frac{1}{2})$	$(11, 12\frac{5}{2})$	$(12, 20\frac{1}{2})$	$(10, 11\frac{3}{2})$	$(12, 12\frac{5}{2})$	$(11, 11\frac{1}{2})$	$(13, 20\frac{1}{2})$	$(11, 11\frac{3}{2})$	$(13, 12\frac{5}{2})$	
$^{40}_{\Lambda}$Ca	3·41	5·69	8·41*	11·83	14·11	16·83*	17·94	20·24	20·32	22·52
	(–)	(–)	(9·00)	(–)	(–)	(17·00)	(–)	(–)	(–)	(23·00)
	$(10, 20\frac{1}{2})$	$(10, 12\frac{5}{2})$	$(11, 12\frac{3}{2})$	$(11, 20\frac{1}{2})$	$(11, 12\frac{5}{2})$	$(12, 12\frac{3}{2})$	$(10, 11\frac{1}{2})$	$(12, 20\frac{1}{2})$	$(10, 11\frac{3}{2})$	$(12, 12\frac{5}{2})$
$^{51}_{\Lambda}$V	7·97*	8·93	11·57	13·33	15·93*	16·90	19·53	21·29	23·89	
	(9·00)	(–)	(–)	(–)	(15·50)	(–)	(–)	(–)	(23·00)	
	$(11, 13\frac{7}{2})$	$(10, 12\frac{3}{2})$	$(10, 20\frac{1}{2})$	$(10, 12\frac{5}{2})$	$(12, 13\frac{7}{2})$	$(11, 12\frac{3}{2})$	$(11, 20\frac{1}{2})$	$(11, 12\frac{5}{2})$	$(13, 13\frac{7}{2})$	
$^{89}_{\Lambda}$Y	6·96*	11·41	11·34	12·78	13·92*	18·29	18·42	19·74	20·68	20·87*
	(7·00)	(–)	(–)	(–)	(14·00)	(–)	(–)	(–)	(–)	(21·00)
	$(11, 14\frac{9}{2})$	$(10, 13\frac{7}{2})$	$(10, 21\frac{3}{2})$	$(11, 21\frac{1}{2})$	$(12, 14\frac{9}{2})$	$(11, 21\frac{3}{2})$	$(11, 13\frac{7}{2})$	$(12, 12\frac{1}{2})$	$(10, 12\frac{3}{2})$	$(13, 14\frac{9}{2})$

$n=1$)

$$E_{nlj}=[2n+l-\tfrac{1}{2}]\hbar\omega_{N_h}+[j(j+1)-l(l+1)-\tfrac{3}{4}]\bar{v}^{ls}_{N_h}$$
$$\times\left[\frac{\pi(n-1)!(l+2)!r_0^3}{\sqrt{2}R^4R_b^{(l+3)}b_{N_h}^{(2l+3)}\Gamma(n+l+\frac{1}{2})}\right]\left\{\binom{n+l-\frac{1}{2}}{n-1}^2\right\}, \tag{8}$$

where $R_b=\dfrac{2R^2+b^2_{N_h}}{2R^2b^2_{N_h}}$, $b_{N_h}=\left[\dfrac{\hbar}{m_N\omega_N}\right]^{1/2}$ and $\bar{v}^{ls}_{N_h}$ is the N_h spin-orbit strength which will be treated as an adjustable parameter. For $n=2$, rest is same as given by (8) but the expression within the curly bracket in (8) is

$$\left\{\binom{n+l-\frac{1}{2}}{n-1}^2-\frac{(l+2)}{R_bb_{N_h}}\binom{n+l-\frac{1}{2}}{n-1}\binom{n+l-\frac{1}{2}}{n-2}+\frac{(l+3)(l+4)}{R_b^2b^2_{N_h}}\binom{n+l-\frac{1}{2}}{n-2}^2\right\}. \tag{9}$$

The oscillator frequency of the N_h is taken to be of the same form as given in the literature for the neutron oscillator frequency [12]

$$\hbar\omega_{N_h}=C_{N_h1}A_c^{-1/3}-C_{N_h2}A_c^{-1}, \tag{10}$$

with $C_{N_h1}=C_{N1}-d_{N_h}$ and $C_{N_h2}=C_{N2}-d_{N_h}$, where C_{N1} and C_{N2} are parameters for the neutron given in the standard literature [12]. These have been obtained by fitting the r.m.s. radius of the nuclei. Since there is no apriori reason to assume that the values of the parameters for N_h and the neutron are same, we take d_{N_h} as an adjustable parameter.

Thus, excitation energy of the Λ-hypernucleus (measured from the ground level) can be written as (for $n_1=n_2=1$)

$$E^*=E^*_\Lambda+E^*_{N_h}$$
$$=[2n_\Lambda+l_\Lambda-2]\hbar\omega_\Lambda+[(2n_1+l_1)-(2n_2+l_2)]\hbar\omega_{N_h}+\bar{v}^{ls}_{N_h}\frac{\pi r_0^3}{\sqrt{2}R^4}$$
$$\times\left[\frac{(n_1-1)!(l_1+2)![j_1(j_1+1)-l_1(l_1+1)-\frac{3}{4}]}{R_b^{(l_1+3)}b_{N_h}^{(2l_1+3)}\Gamma(n_1+l_1+\frac{1}{2})}\left\{\binom{n_1+l_1-\frac{1}{2}}{n_1-1}^2\right\}\right.$$
$$\left.-\frac{(n_2-1)!(l_2+2)![j_2(j_2+1)-l_2(l_2+1)-\frac{3}{4}]}{R_b^{(l_2+3)}b_{N_h}^{(2l_2+3)}\Gamma(n_2+l_2+\frac{1}{2})}\left\{\binom{n_2+l_2-\frac{1}{2}}{n_2-1}^2\right\}\right]. \tag{11}$$

For either n_1 or $n_2=2$, their respective expression within the curly bracket, in the above equation, should be replaced by (9). At the expense of some additional small error, we could have used, for the N_h spin-orbit splitting, the simpler formula for the nucleon spin-orbit splitting given in the literature [13]

$$\Delta E^{ls}=1{\cdot}4V^{ls}_N(l+\tfrac{1}{2})A_c^{-2/3}. \tag{12}$$

Thus, the excitation energy E^* (using (12)) may be written as:

$$E^*=[2n_\Lambda+l_\Lambda-2]\hbar\omega_\Lambda+[(2n_1+l_1)-(2n_2+l_2)]\hbar\omega_{N_h}+0{\cdot}7V^{ls}_{N_h}A_c^{-2/3}$$

$$\times [[l_1 \delta_{j_1(l_1-\frac{1}{2})} - (l_1+1)\delta_{j_1(l_1+\frac{1}{2})}] \times (1-\delta_{l_1 0}) - [l_2 \delta_{j_2(l_2-\frac{1}{2})} - (l_2+1)\delta_{j_2(l_2+\frac{1}{2})}] \times (1-\delta_{l_2 0})]. \qquad (11')$$

3. Results and discussion

Using (11), we carry out χ^2-fitting of all the available excitation data over a large range of mass numbers assuming a certain fixed percentage error for all the nuclei considered. The χ^2 is defined to be the quantity

$$\sum_i \left[\frac{Q^{\exp}(i) - Q^{\rm th}(i)}{\Delta Q^{\exp}(i)} \right]^2,$$

where the experimentally quoted value is taken to be $Q_i^{\exp} \pm \Delta Q_i^{\exp}$, the summation extending to all the data points. $\Delta Q_i^{\exp}$ is chosen to be an average percentage error of 5% for all the nuclei. This seems quite realistic. Here the value of χ^2 is not to be taken as justifying the above picture which is over-simplistic, as already indicated. The value of the simulated χ^2 should be taken only to convince that ours is a semi-quantitative formula for the spectral energy spacings. Our input data consist of the spectra of the hypernuclei, ${}^{6}_{\Lambda}$Li, ${}^{7}_{\Lambda}$Li, ${}^{9}_{\Lambda}$Be, ${}^{12}_{\Lambda}$C, ${}^{13}_{\Lambda}$C, ${}^{16}_{\Lambda}$O, ${}^{27}_{\Lambda}$Al, ${}^{28}_{\Lambda}$Si, ${}^{32}_{\Lambda}$S, ${}^{40}_{\Lambda}$Ca, ${}^{51}_{\Lambda}$V, ${}^{89}_{\Lambda}$Y and ${}^{209}_{\Lambda}$Bi.

Keeping C_{N1} and C_{N2} fixed at their respective values given in the standard literature [12], χ^2-fitting of all the available hypernuclear data treating $C_{\Lambda 1}$, $C_{\Lambda 2}$, d_{N_h} and $\bar{v}^{ls}_{N_h}$ as adjustable has been carried out. The results are given in table 1. The best fit parameters, for $\Delta Q_i^{\exp}$ taken as 5%, are $C_{\Lambda 1} = 38{\cdot}684$ MeV, $C_{\Lambda 2} = 34{\cdot}411$ MeV, $d_{N_h} = -19{\cdot}479$ MeV and $\bar{v}^{ls}_{N_h} = -4{\cdot}467$ MeV, giving total $\chi^2 = 49{\cdot}78$ for thirty two data points. With our limitation it is justified to regard this as a semi-quantitative fit. If we use (11′) we get more or less similar results, with $\chi^2 = 59{\cdot}86$. So, we may use the simpler formula, at the expense of a small increase in the total χ^2. The best fit parameters in this case are $C_{\Lambda 1} = 38{\cdot}356$ MeV, $C_{\Lambda 2} = 33{\cdot}877$ MeV, $d_{N_h} = 25{\cdot}418$ MeV and $V^{ls}_{N_h}$ 18·037 MeV. Our spin-orbit splitting for N_h is comparable to that for ordinary nucleons in conventional nuclear physics obtained from nucleon–nucleus scattering, etc.

We employ Bouyssy's notation for the ground state only for the purpose of getting the level-spacing with respect to the ground state energy and then the successive excited states are obtained by varying the quantum numbers. The task is easier if the somewhat more approximate formula (11′) is employed.

The results are given in table 1. For the hypernuclei studied in this work, all the excitation energies up to the highest value observed for that nucleus are calculated. Transitions marked by an asterisk do not involve any N_h transition in our picture of the oscillators. It is seen that many levels have not been observed in the experimental spectra. Such a situation is well-known. We may point out that relatively a smaller number of the starred states is not observed. Possibly some of the unobserved states have merged out to produce the neighbouring state observed in the experiment. We note that in most cases, the first excited state arises from pure Λ-excitations, but in two of our cases (i.e. ${}^{16}_{\Lambda}$O and ${}^{40}_{\Lambda}$Ca), the first excited state arises from neutron-hole excitation. In ${}^{16}_{\Lambda}$O the hole transition is $1p_{1/2} \to 1p_{3/2}$ and in ${}^{40}_{\Lambda}$Ca it is $1d_{3/2} \to 2s_{1/2}$. In passing, we may mention that the simulated χ^2 would be reduced by about 4 if the observed excitation of 21·5 MeV for ${}^{7}_{\Lambda}$Li is identified with the third rather than the second excited state.

In ${}^{40}_{\Lambda}$Ca, 15 excited states have been calculated (only 10 are shown in the table). The energies of the 15th predicted and the observed excited states are 30·94 MeV and 31·50 MeV, respectively. In ${}^{209}_{\Lambda}$Bi, only one state has been observed and on the basis of our calculations it is the 38th excited state. Remembering that the energy resolution is about 3 MeV, it seems that almost all of these states have been averaged out because of their close spacing. Therefore, such levels as arise from spin-flip etc. or where the single-particle states are too close are all washed out giving an average distribution in accordance with observations. Further, since the deep hole states are highly excited, these have a large width. Consequently, these are almost completely washed out giving a smooth cross-section. Thus, whereas in (K^-, π^-) reaction giving ${}^{40}_{\Lambda}$Ca some structure corresponding to rather deep holes is observed, the cross-section is completely smoothed out where the same nucleus is produced in (π^+, K^+) reaction.

Since in the region of heavy nuclei, for most states, the spacing is very small, and since our formula gives the energy spacings with an error of a little over 1 MeV, it follows that the formula would be useful for nuclei ranging from light to medium-heavy but generally not as useful for very heavy nuclei like Bi. In the region where the neutron sub-shells are bunched rather closely or those hypernuclei of the $1p$-shell in which the outer-most neutrons are in the $1p$-shell, one should expect that the neutron-hole excitation would occur at a lower energy than the Λ-excitation. So, one expects that for these hypernuclei, the first excited state arises from neutron-hole excitation only. However, for reasons given above, this only makes sense if the hypernucleus is not too heavy. The situation in ${}^{16}_{\Lambda}$O and ${}^{40}_{\Lambda}$Ca is according to the above expectations.

4. Prediction of pure Λ-excitations from some hypernuclear ground state calculations

For states of pure Λ-excitations, i.e. those in which the N_h is in the topmost occupied orbital, the Λ-binding energy in an oscillator orbit (nl), denoted as $(B_\Lambda)_{nl}$, may be written as

$$(B_\Lambda)_{nl} = (B_\Lambda)_g + [2 - (2n + l)]\hbar\omega_\Lambda, \tag{13}$$

where $(B_\Lambda)_g$ denotes the ground state Λ-binding energy and $\hbar\omega_\Lambda$ is the Λ-oscillator frequency.

It can be easily seen that the condition for having only one bound state in the Λ-nucleus potential is

$$(B_\Lambda)_g < \hbar\omega_\Lambda. \tag{14}$$

In microscopic calculations, the wave function may be regarded as a superposition of several oscillator states. However, if crudely, the Λ is assigned a pure oscillator state of a certain frequency ω_Λ, the latter may be obtained by equating the expectation value of the kinetic energy to $\frac{3}{4}\hbar\omega_\Lambda$. Variational Monte Carlo calculations [14, 15] of B_Λ, from which ω_Λ can be extracted in the above manner, are available to us for ${}^{3}_{\Lambda}$H, ${}^{4}_{\Lambda}$H, ${}^{4}_{\Lambda}$He, ${}^{5}_{\Lambda}$He and ${}^{9}_{\Lambda}$Be. It can be easily seen from the value of our parameters that the distance between the neighbouring N_h-shells is always larger than that between the Λ-shells. Therefore, it is predicted that the first excited state in all $1s$-shell nuclei shall be a pure Λ-excitation. In $1s_{1/2}$-shell, N_h-transition seems to have no meaning. We can easily see that for the $1p$-shell nuclei having nucleons up to $1p_{3/2}$-shell only but not in the $1p_{1/2}$-shell the only N_h-transition can be from $1p_{3/2} \rightarrow 1s_{1/2}$. From the simpler formula (11′) it can be easily seen that for all these nuclei the N_h-transition energy is certainly larger than the pure Λ-excitation energy. Further, for $1p$-shell nuclei having nucleons in the $1p_{1/2}$-shell as well, the lowest N_h-transition is $1p_{1/2} \rightarrow 1p_{3/2}$ and this energy is also larger than that of lowest pure Λ-excitation. Therefore, the lowest excitation for all the nuclei is the Λ-excitation i.e. the excitation energy is $\hbar\omega_\Lambda$.

However, the excitation data are available only in respect of ${}^{9}_{\Lambda}$Be which is a $1p$-shell hypernucleus. Thus, the first excited state arise from pure Λ-excitation. The value of $\hbar\omega_\Lambda$ extracted from microscopic calculations satisfies inequality (14) above in accordance with the observation in that this nucleus has only one bound excited state.

Results of $1p$-shell phenomenological calculations of Shoeb and Rahman Khan [16], using oscillator wave functions, and employing, in addition to the two-body ΛN force also a three-body ΛNN force, are consistent with the above predictions in all the cases for which the data are available and for which the calculations were carried out. Two of the four gaussian ΛN potentials, along with the three-body ΛNN force, employed by Ansari *et al* [17] are also consistent with the predictions. A comparison with the work of Mujib *et al* [18] brings out the significance of three-body ΛNN force in this context.

The predicted values of $\hbar\omega_\Lambda$ from empirical relations, $\hbar\omega_\Lambda = 60\, A_c^{-2/3}$, given by Auerbach and Gal [19]; $\hbar\omega_\Lambda = 27\, A_c^{-1/3}$, given by Bouyssy [4] and $\hbar\omega_\Lambda = C_{\Lambda 1} A_c^{-1/3} - C_{\Lambda 2} A_c^{-2/3}$ used by us are not very different for medium and heavy nuclei, but are markedly different for light nuclei. It, therefore, follows that spectral studies of light nuclei would help in discriminating between the various formulae for $\hbar\omega_\Lambda$.

5. Conclusion

A semi-quantitative formula, giving the energy spacings of the Λ-hypernuclear excitation data over a large mass number range, is presented. However, the simple picture of the oscillators employed for the purpose of giving the semi-empirical formula is not close to reality, although it might well turn out that by inclusion of a residual interaction and some alteration in the value of the parameters, it might get us quite close to the more realistic situation. We entertain this hope because the agreement achieved cannot be entirely fortuitous, but such conjectures are not crucial for the justification of the work reported here. The picture is used only to obtain a semi-empirical formula to account for the observed spectral energy spacings. Our formula seems to be satisfactory for light and medium nuclei and perhaps not useful for heavier ones.

The idea of harmonic excitation allows us to make reasonable predictions about the excited states from microscopic or phenomenological calculations of hypernuclear ground states.

It is pointed out that study of light nuclei is likely to be more useful for discriminating between the various empirical formulae for $\hbar\omega_\Lambda$. The same should apply to formulae for $\hbar\omega_{N_h}$. Our formulae show that the level spacings due to pure Λ-excitations as well as due to N_h-transition get smaller and smaller with increasing mass number. This is, indeed, what one expects on general grounds and this is responsible for the smearing out of the spectra rendering formulae like ours almost useless.

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