PRAMANA
— journal of
physics

© Printed in India

Vol. 41, No. 6,
December 1993
pp. 509–513

# Quantum instability for a wave packet kicked periodically by a quadratic potential

A K SIKRI and M L NARCHAL
Department of Physics, Punjabi University, Patiala 147 002, India

MS received 23 April 1993; revised 16 August 1993

**Abstract.** The quantal behaviour of wave packet periodically kicked by a quadratic potential has been investigated using Floquet theory. The wave function of the particle after $N$ kicks has been determined. This wave function has been utilized to obtain the packet width, energy and loss of memory as a function of the number of kicks. It has also been shown that the free particle wave packet quasienergy spectrum makes a transition from discrete to absolutely continuous at $\varepsilon T = 2$. The behaviour for $\varepsilon T > 2$ is quantum mechanically irregular.

**Keywords.** Quantum recurrence; quantum instability; chaos.

**PACS No.** 05·45

## 1. Introduction

There has been tremendous interest in recent years in the study of quantum systems with time dependent Hamiltonians which show instability or chaos in the classical limit. A lot of literature [1–10] has thus appeared, pioneering among them are those of Casati *et al* [1], Berry [2], Fishman *et al* [3] and Bellissard [6]. The principal motivation behind these studies is to determine as to how far the quantum systems mimic their classical counterparts. For example if the classical behaviour goes from regular to irregular as a certain system parameter crosses a critical value, does the quantum behaviour do so? The system which has been most studied is the quantum kicked rotator because of its link with the Anderson localization in disordered solids [3, 4]. It has been shown [11, 12] that the system is capable of showing both quantum mechanically recurrent as well as nonrecurrent behaviour depending upon whether the quasienergy states are localized or extended. A situation in which the quasienergy states are extended and the energy is unbounded is a quantally unstable situation without chaos whereas if the energy growth is diffusive the situation may be termed as quantum mechanically chaotic. Evidence for short time diffusive growth of energy exists in a quantum kicked rotator whereas the long time growth is not diffusive [11]. Thus as far as the present understanding goes, one has only obtained short time quantum chaos which gets scuttled by quantum effects with the passage of time.

Besides the quantum kicked rotator, several other problems have been studied in which the system shows nonrecurrent behaviour [8, 13]. Casati *et al* [11] have evolved a criterion to test whether the nonrecurrent behaviour is irregular or chaotic. In terms of this criterion if the quasienergy spectrum is absolutely continuous then the system behaviour is irregular whereas if it is singularly continuous the behaviour is chaotic.

The simplest time dependent system which shows classical instability is that of a free particle kicked periodically by a potential quadratic or higher in space. Classical studies of this problem have been reported by various workers [8, 14, 15]. There is no exact quantum mechanical solution of this problem which gives a unified view of regular as well as irregular behaviour. This paper presents an exact analytical solution to this problem. In § 2 we obtain the wave function of the particle after $N$ kicks and utilize this wave function in § 3 to obtain the packet width, energy and loss of memory as a function of the number of kicks. The results are summarized in § 4.

## 2. Wave function of a system after $N$ kicks

Consider a free particle of mass $m$ described by the time dependent Hamiltonian

$$H = H_0 + \frac{1}{2} m\varepsilon^2 x^2 \sum_n \delta(t - nT) \tag{1}$$

where $H_0 = p^2/2m$ is the free particle Hamiltonian $\varepsilon$, determines the strength of the kick and $T$ is time of kick. The evolution of the system is governed by one step Floquet operator

$$U = \exp(-(i/\hbar) H_0 T)\exp(-im\varepsilon^2 x^2 T/2\hbar). \tag{2}$$

Following Blumel *et al* [13] this operator may be written as

$$U = \exp(-iGT/\hbar) \tag{3}$$

where

$$G = \lambda\left[\frac{p^2}{2m} + \frac{1}{2} m\varepsilon^2 x^2 + \frac{\varepsilon^2 T}{4}(xp + px)\right] \tag{4}$$

$$\lambda = \sin^{-1}(r\varepsilon T)/r\varepsilon T, \tag{5}$$

$$r^2 = 1 - \varepsilon^2 T^2/4. \tag{6}$$

Equation (6) shows that $r$ is real if $\varepsilon T < 2$ whereas it becomes imaginary for $\varepsilon T > 2$. Thus the spectrum of operator $G$ will be determined by the value of $\varepsilon T$.

Let $|\psi_0\rangle$ be the wave function of the free particle at time $t = 0$. Then its wave function after $N$ kicks is

$$|\psi_N\rangle = U^N \quad |\psi_0\rangle. \tag{7}$$

Using (3) it takes the form

$$|\psi_N\rangle = \exp(-iNGT/\hbar) \quad |\psi_0\rangle. \tag{8}$$

In the position representation it may be written as

$$\psi_N(x) = \int_{-\infty}^{\infty} \langle x|\exp(-iNGT/\hbar)|x'\rangle \quad \psi_0(x')\mathrm{d}x'. \tag{9}$$

We represent the free particle by the wave packet of width $\sigma$ at $t = 0$ by

$$\psi_0(x') = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp((-x'^2/4\sigma^2) + ikx'). \tag{10}$$

Thus the expression for $\psi_N(x)$ becomes

$$\psi_N(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \int_{-\infty}^{\infty} \langle x|\exp(-iNGT/\hbar)|x'\rangle \exp(-x'^2/4\sigma^2) + ikx'\,\mathrm{d}x'. \tag{11}$$

This integral can be evaluated by proceeding on the lines adopted by Blumel *et al* [13]. Thus the wave function after $N$ kicks takes the form

$$\psi_N(x) = \left(-\frac{v}{2\pi\sigma^2}\right)^{1/4} \exp\{(-1/4\sigma^2)(vx^2 + (kv\sin(\eta)x/br) + (k^2 v\sin^2\eta/4b^2r^2))$$
$$\times \exp(-i(bdx^2 - (kav\sin(\eta)x/r) + \tfrac{1}{2}\tan^{-1}(4ab\sigma^2) - (k^2 av\sin^2\eta/4br^2))). \tag{12}$$

where

$$b = m\varepsilon/2\hbar, \tag{13}$$

$$v = r^2\left[\frac{\sin^2\eta}{16b^2\sigma^4} + \left(r\cos\eta + \frac{\varepsilon T}{2}\sin\eta\right)^2\right]^{-1}, \tag{14}$$

$$\eta = N\sin^{-1}(\varepsilon rT), \tag{15}$$

$$d = a(1-v) - \varepsilon T, \tag{16}$$

$$a = r\cot\eta + \frac{\varepsilon T}{2}. \tag{17}$$

## 3. Packet width, energy and loss of memory

The width of wave packet after $N$ kicks is given by

$$\delta_N = [\langle x^2\rangle_N - \langle x\rangle_N^2]^{1/2}, \tag{18}$$

where $\langle\ \rangle_N$ represents the expectation value with respect to wave function $\psi_N$. Using (12) it takes the form

$$\delta_N = \frac{\sigma}{r}\left[\frac{\sin^2\eta}{16b^2\sigma^4} + \left(r\cos\eta + \frac{\varepsilon T}{2}\sin\eta\right)^2\right]^{1/2}. \tag{19}$$

This equation shows that for $\varepsilon T < 2$ the width oscillates in time with frequency $|\sin^{-1}(\varepsilon rT)|/T$. This is the manifestation of quantum recurrence. For $\varepsilon T > 2$, $\eta$ becomes imaginary and $\sin\eta$ and $\cos\eta$ become $i\sinh\eta$ and $\cosh\eta$ respectively. So the width grows exponentially with time scale $T/|\sin^{-1}(\varepsilon rT)|$. This may be compared with the natural spreading of the wave packet which takes place even in the absence of kicks. The natural width of the wave packet at any time $t$ can be obtained be letting $\varepsilon \to 0$ in (19). This gives

$$\delta_0 = \sigma\left(1 + \frac{\hbar^2 t^2}{4m^2\sigma^4}\right)^{1/2}, \tag{20}$$

where $t = NT$. We observe that the increase of width is much faster in the presence of kicks as compared to natural spreading. This increase is determined by the product of kicking strength and time of kick.

The energy of free particle after $N$ kicks is given by

$$E_N = \langle \psi_N | H_0 | \psi_N \rangle. \tag{21}$$

Using (12) we obtain

$$E_N = \frac{\hbar^2}{2m}\left[\frac{v}{4\sigma^2} + \frac{2b^2 d^2 \sigma^2}{v} + \frac{k^2 \sin^2 \eta}{r^2}(av + d)^2\right]. \tag{22}$$

For $\varepsilon T < 2$, the energy oscillates showing quantum recurrence whereas for $\varepsilon T > 2$, the energy shows an exponential growth with a time scale proportional to the kicking strength.

The loss of memory after $N$ kicks is determined by the absolute value of overlap integral $\langle \psi_N | \psi_0 \rangle$. That is

$$C_N = |\langle \psi_N | \psi_0 \rangle|. \tag{23}$$

Using (10) and (12) we obtain

$$C_N = \left(\frac{4v}{(v+1)^2 + 8b^2\sigma^2 d^2}\right)^{1/4} e^{\beta}, \tag{24}$$

where

$$\beta = \frac{k^2}{8D\sigma^2}\left[\frac{v^2 \sin^2 \eta}{r^2}\left\{(v+1)\left(\frac{1 - 8a^2 b^2 \sigma^2}{8b^2\sigma^2}\right) - 2ad - \frac{D}{b^2 v}\right\} + \frac{2v \sin \eta}{r}(2a - \varepsilon T) - (v+1)\right]. \tag{25}$$

$$D = \left(\frac{v+1}{4\sigma^2}\right)^2 + b^2 d^2. \tag{26}$$

We find that $C_N$ is a periodic function of $N$ for $\varepsilon T < 2$ which shows that any loss of memory which may take place initially is subsequently recovered. This is again a manifestation of recurrence. For $\varepsilon T > 2$ we find that

$$\operatorname*{Lt}_{N \to \infty} C_N = 0. \tag{27}$$

The system completely forgets about its intial state with the passage of time.

The probability that the free particle makes a transition from initial state $|p'\rangle$ of $H_0$ to final state $|p\rangle$ after $N$ kicks is proportional to the square of the probability amplitude. That is

$$P_{p' \to p} \propto |\langle p | U^N | p' \rangle|^2. \tag{28}$$

For $\varepsilon T > 2$ the spectrum of $U$ is continuous. In order that $U$ possesses absolutely continuous spectrum [12] it is necessary that

$$\operatorname*{Lt}_{N \to \infty} P_{p' \to p} = 0 \quad \text{for all } p \text{ and } p'. \tag{29}$$

Using (3) the expression for probability becomes

$$P_{p' \to p} \propto |\langle p | \exp(-iNGT/\hbar) | p' \rangle|^2. \tag{30}$$

Following Blumel *et al* [13] it can be shown that

$$\operatorname*{Lt}_{N\to\infty} P_{p'\to p}=0. \tag{31}$$

Thus we conclude that the spectrum of Floquet operator for free particle in region $\varepsilon T>2$ is absolutely continuous. This is an indication of a quantum mechanically irregular behaviour.

## 4. Conclusions

The exact analytical solution of a free particle wave packet subject to a sequence of $\delta$ function quadratic kicks has been obtained. There is only one parameter $\varepsilon T$ which decides the behaviour of the system. The quasienergy spectrum makes a transition from discrete to absolutely continuous at $\varepsilon T=2$. Since the spectrum is absolutely continuous for $\varepsilon T>2$, the system may be described as quantum mechanically irregular in this region.

An explicit expression for the energy after an arbitrary number of kicks has been obtained. The system shows markedly different behaviour on the two sides of transition point. The energy of the system for $\varepsilon T<2$ is bounded and this agrees with the corresponding classical predictions. For $\varepsilon T>2$ there is exponential growth of energy with the increase in the number of kicks. This is a manifestation of irregular behaviour of the system. Thus we observe that the system faithfully imitates the classical behaviour.

Besides energy, the width of the wave packet and overlap integral behave differently across the transition point. For $\varepsilon T>2$ the packet width grows exponentially with time but it oscillates in time for $\varepsilon T<2$. The overlap integral, which is a measure of the loss of memory of initial state, is a periodic function of time for $\varepsilon T<2$ whereas it decays to zero with the passage of time for $\varepsilon T>2$. This means that for regular behaviour there is a temporary loss of memory which is subsequently recovered whereas for irregular behaviour there is a permanent loss of memory.

## References

[1] G Casati, B V Chirikov, F M Izrailev and J Ford, in *Stochastic behaviour in classical and quantum Hamiltonian systems*, edited by G Casati and J Ford, Lecture Notes in Physics (Springer-Berlin, 1979) vol. 93
[2] M V Berry, *Proc. of Les-Houches Summer School*, edited by G Iooss, R H G Helleman and R Stora (North Holland, Amsterdam, 1983)
[3] S Fishman, D R Grempel and R E Prange, *Phys. Rev. Lett.* **49**, 509 (1982)
[4] D R Grempel, R E Prange and S Fishman, *Phys. Rev.* **A29**, 1639 (1984)
[5] T Hogg and B A Huberman, *Phys. Rev.* **A28**, 22 (1983)
[6] J Bellissard, in *Trends and developments in the eighties*, edited by S Albeverio and Ph. Blanchard (World Scientific, Singapore, 1985)
[7] J V Jose, *Phys. Rev. Lett.* **56**, 290 (1986)
[8] A Cohen and S Fishman, *Int. J. Mod. Phys.* **2**, 103 (1988)
[9] Petr. Seba, *Phys. Rev.* **A41**, 2306 (1990)
[10] M Combescure, *J. Stat. Phys.* **59**, 679 (1990)
[11] G Casati, J Ford, I Guarneri and F Vivaldi, *Phys. Rev.* **A34**, 1413 (1986)
[12] B Milek and P Seba, *Phys. Rev.* **A42**, 3213 (1990)
[13] R Blumel, R Meir and U Smilansky, *Phys. Lett.* **A103**, 353 (1984)
[14] M V Berry, N L Balazs, M Tabor and A Voros, *Ann. Phys.* (NY) **122**, 26 (1979)
[15] H J Korsch and M V Berry, *Physica* **D3**, 627 (1981)