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Integration method for the Dirac equation

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Abstract. An integration method for the Dirac equation is proposed. The method, based on diagonalization, reduces the problem to one of integration of independent second-order differential equations.

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We start from the Dirac equation presented in the form (Chandrasekhar [1])

$$\mathscr{H}_1 = \sigma^i_{AB'} i\partial_i V^A - m\bar{W}_{B'} = 0,$$
$$\mathscr{H}_2 = \sigma^i_{AB'} i\partial_i W^A - m\bar{V}_{B'} = 0$$

and we carry out the next transformations:

a) $\mathscr{H}_1 = 0 \Rightarrow \sigma_{2DB'}(\varepsilon^{B'C'}\sigma^i_{AC'} i\partial_i V^A - m\bar{W}^{B'}) = 0,$

b) $\mathscr{H}_2 = 0 \Rightarrow (\varepsilon^{BC}\bar{\sigma}^i_{A'C} i\partial_i \bar{W}^{A'} + mV^B) = 0 \Rightarrow$

$\Rightarrow \sigma_{2D'B}(\varepsilon^{BC}\bar{\sigma}^i_{A'C} i\partial_i \bar{W}^{A'} + mV^B) = 0,$

c) $i\partial_i \Rightarrow i\partial_i + A_i$

(A_i is electromagnetic potential).

Then Dirac equation can be presented in a spinor form:

$$\hat{\mathscr{H}}\hat{\Psi} = \left[\begin{pmatrix} \hat{0} & \hat{\Box}_1 \\ \sigma_2\hat{\Box}_2\sigma_2 & \hat{0} \end{pmatrix} + im\begin{pmatrix} \sigma_2 & \hat{0} \\ \hat{0} & -\hat{\sigma}_2 \end{pmatrix}\right]\begin{pmatrix} V^A \\ \bar{W}^A \end{pmatrix} = 0, \quad (1)$$

$$\hat{\Box}_1 = \sqrt{2}(\hat{1}P_1 + \hat{4}P_0 - \hat{2}P_3 - \hat{3}P_2), \quad \Box_2 = \sqrt{2}(\hat{1}P_0 + \hat{4}P_1 + \hat{2}P_3 + \hat{3}P_2),$$

$$P_i = i\partial_i + A_i, \quad \partial_i = \partial/\partial u^i, \quad \sqrt{2}u^0 = (x^0 + x^1), \quad \sqrt{2}u^1 = (x^0 - x^1),$$

$$\sqrt{2}u^2 = (x^2 + ix^3), \quad u^3 = \overline{u^2}, \quad \hat{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\hat{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{4} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\sigma_i = (\hat{2} + \hat{3}), \sigma_2 = i(\hat{2} - \hat{3}), \sigma_3 = (\hat{1} - \hat{4})$, x^i are Descartes coordinates, V^A, W^A are spinor fields. Propose the first order linear differential operator $\hat{A}$, the matrix $\hat{\Omega}$ and the linear diagonal second order differential operator $\hat{\mathscr{L}}$ satisfying the condition

$$\hat{\mathscr{H}}\hat{A} = \hat{\Omega}\hat{\mathscr{L}} \tag{2}$$

exist. (The problem for the Dirac–Fock–Ivanenko squared equation was studied by Bagrov and Obukhov [2]. Then the integration problem for the Dirac equation may be reduced to one of integration of independent second-order differential equations:

$$\hat{\mathscr{L}}\hat{\Phi} = 0 \Rightarrow \hat{\Psi} = \hat{A}\hat{\Phi}. \tag{3}$$

One can show that operator $\hat{A}$ has a form

$$\hat{A} = \hat{H}\hat{\Omega} + \hat{\Gamma},$$

$$\hat{H} = \begin{pmatrix} \hat{0} & \sigma_2\hat{\Box}_1\sigma_2 \\ \hat{\Box}_2 & \hat{0} \end{pmatrix} + im\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{4}$$

Note that in this case the condition $\det \hat{\Omega} \neq 0$ is not considered. Let us denote

$$\hat{\Omega} = \begin{pmatrix} \hat{B} & \hat{B}^* \\ \hat{\bar{Y}} & \hat{\bar{Y}}^* \end{pmatrix}, \quad \hat{\Gamma} = \begin{pmatrix} \hat{\bar{\tau}} & \hat{\bar{\tau}}^* \\ \hat{\gamma} & \hat{\gamma}^* \end{pmatrix}, \quad \hat{\Phi} = \begin{pmatrix} -\Phi_1, & -\Phi_2 \\ \Phi_0, & \Phi_1 \end{pmatrix},$$

$$\hat{\mathscr{L}} = \Box + 2i\left[\begin{pmatrix} \hat{R} & \hat{0} \\ \hat{0} & \hat{R}^* \end{pmatrix}P_0 + \begin{pmatrix} \hat{T} & \hat{0} \\ \hat{0} & \hat{T}^* \end{pmatrix}P_1 - \begin{pmatrix} \hat{K} & \hat{0} \\ \hat{0} & \hat{K}^* \end{pmatrix}P_2 - \right.$$

$$\left. - \begin{pmatrix} \hat{N} & \hat{0} \\ \hat{0} & \hat{N}^* \end{pmatrix}P_3 - \begin{pmatrix} \hat{P} & \hat{0} \\ \hat{0} & \hat{P}^* \end{pmatrix}\right],$$

$$\Phi_0 = F_{02}, \quad \Phi_2 = (F_{01} + F_{32})/2, \quad \Phi_3 = F_{31},$$

$\Box$ is the Klein–Gordon–Fock operator. Then (2) takes the form

$$\begin{aligned} &\hat{B}_{,0} + \hat{1}\hat{\gamma} = \hat{B}\hat{T}, &\quad &\hat{Y}_{,0} + \hat{1}\hat{t} = \hat{Y}\hat{\hat{T}}, \\ &\hat{B}_{,1} + \hat{4}\hat{\gamma} = \hat{B}\hat{R}, &\quad &\hat{Y}_{,1} + \hat{4}\hat{t} = \hat{Y}\hat{\hat{R}}, \\ &\hat{B}_{,2} + \hat{2}\hat{\gamma} = \hat{B}\hat{N}, &\quad &\hat{Y}_{,2} + \hat{3}\hat{t} = \hat{Y}\hat{\hat{N}}, \\ &\hat{B}_{,3} + \hat{3}\hat{\gamma} = \hat{B}\hat{K}, &\quad &\hat{Y}_{,3} + \hat{2}\hat{t} = \hat{Y}\hat{\hat{K}}, \end{aligned} \tag{5}$$

$$\hat{B}_{,01} - \hat{B}_{,23} + \mu\sigma_2\hat{t} - i\hat{\Phi}\hat{B} + \hat{1}\hat{\gamma}_{,1} - \hat{2}\hat{\gamma}_{,3} - \hat{3}\hat{\gamma}_{,2} + \hat{4}\hat{\gamma}_{,0} = i\hat{B}\hat{P}, \tag{6}$$

$$\hat{Y}_{,01} - \hat{Y}_{,23} + \mu\sigma_2\hat{\bar{\gamma}} - i\hat{\Phi}\hat{Y} + \hat{1}\hat{t}_{,1} - \hat{2}\hat{t}_{,3} - \hat{3}\hat{t}_{,2} + \hat{4}\hat{t}_{,0} = -i\hat{Y}\hat{\hat{P}}, \tag{7}$$

$$\mu = m/\sqrt{2}.$$

The quantities with the asterisks have the same form, hence they have not been considered here. We present all solutions of (5)

$$\begin{aligned} \text{I.}\ &\hat{B} = \hat{u}\hat{a} + \hat{k}, \quad \hat{Y} = \hat{u}\hat{b} + \hat{n}, \quad \hat{T} = \hat{R} = \hat{K} = \hat{N} = 0, \\ &\hat{\gamma} = -\hat{a}, \quad \hat{t} = -\hat{b}, \quad \hat{u} = u^0\hat{1} + u^2\hat{2} + u^3\hat{3} + u^1\hat{4}. \end{aligned} \tag{8}$$

$$\begin{aligned} \text{II.}\ &\hat{B} = \hat{2} + \hat{4}\exp Q + \hat{u}(a_1\hat{1} + a_3\hat{3} + \hat{4}), \quad \hat{Y} = \hat{u}(b_1\hat{1} + b_3\hat{3}) + n_1\hat{1} + n_3\hat{3}. \\ &\hat{\gamma} = -a_1\hat{1} + Q_{,0}\hat{2} - a_3\hat{3} - Q_{,1}\hat{4}, \quad \hat{t} = -(b_1\hat{1} + b_3\hat{3}). \end{aligned} \tag{9}$$

The function Q satisfies the set of equations

$$Q_{,0} + (\exp Q)_{,3} = 0,$$
$$Q_{,2} + (\exp Q)_{,1} = 0.$$

$$\text{III.}\ \hat{B} = \hat{1} + Q\hat{3},\quad \hat{Y} = \hat{2} + J\hat{4},\quad \hat{\gamma} = \hat{1}Q_{,0}/Q - \hat{3}Q_{,1},\quad \hat{\tau} = \hat{2}J_{,0}/J - \hat{4}J_{,1}. \tag{10}$$

The functions Q, J satisfy the sets of equations

$$\begin{matrix} Q_{,3} + Q_{,0}/Q = 0 \\ Q_{,1} + Q_{,2}/Q = 0 \end{matrix},\quad \begin{matrix} J_{,1} + J_{,3}/Q = 0 \\ J_{,2} + J_{,0}/Q = 0 \end{matrix}.$$

The matrices

$$\begin{aligned} \hat{a} &= a_1\hat{1} + a_2\hat{2} + a_3\hat{3} + a_4\hat{4}, \quad \hat{b} = b_1\hat{1} + b_2\hat{2} + b_3\hat{3} + b_4\hat{4}, \\ \hat{k} &= k_1\hat{1} + k_2\hat{2} + k_3\hat{3} + k_4\hat{4}, \quad \hat{n} = n_1\hat{1} + n_2\hat{2} + n_3\hat{3} + n_4\hat{4} \end{aligned} \tag{11}$$

are constants.

Let us show that due to the sets (6) and (7), solutions II, III reduce to solution I. Indeed using solution II one can present the set (7) to the form:

$$\mu\sigma_2(-a_1\hat{1} + Q_{,0}\hat{2} - a_3\hat{3} - Q_{,1}\hat{4}) + i(-\Phi_1\hat{1} - \Phi_2\hat{2} + \Phi_0\hat{3} + \Phi_1\hat{4})$$
$$(Y_1\hat{1} + Y_3\hat{3}) + i(Y_1\hat{1} + Y_3\hat{3})(\bar{\mathscr{P}}_1\hat{1} + \mathscr{P}_2\hat{2}) = 0. \tag{12}$$

Hence $Q_{,0} = Q_{,1} = 0$ and from set (9) it follows that $Q = \text{constant}$. The same approach for solution III gives the result: Q, $J = \text{constant}$.

That is why we believe that $\hat{B}$, $\hat{Y}$, $\hat{T}$, $\hat{R}$, $\hat{K}$, $\hat{N}$, $\hat{\gamma}$, $\hat{\tau}$ are given by correlations (8). The sets (6), (7) now have the form

$$\hat{\Phi}\hat{B} + i\mu\sigma_2\hat{\hat{b}} = \hat{\Phi}\hat{P}, \tag{13}$$

$$\hat{\Phi}\hat{Y} + i\mu\sigma_2\hat{a} = \hat{Y}\hat{\hat{P}}. \tag{14}$$

Obviously the transformations rule for the matrices is valid:

$$\begin{cases} \hat{k}' = \hat{\Lambda}\hat{k}, \quad \hat{n}' = \hat{\Lambda}\hat{n}, \quad \hat{a}' = \hat{\hat{l}}\hat{a}, \quad \hat{b}' = \hat{\hat{l}}\hat{b}, & (15) \\ \det\hat{\Lambda} = \det\hat{l} = 1, \quad \hat{\Lambda}\hat{l} = \hat{I}. & (16) \end{cases}$$

Moreover, the matrices $\hat{B}$, $\hat{Y}$ can be acted by transformation

$$\hat{B}' = \hat{J}^{-1}\hat{B}\hat{J}.$$

Provided that operator $\hat{L}'$

$$\hat{L}' = \hat{J}^{-1}\hat{L}\hat{J}$$

remains diagonal.

All these transformations form a group. Using the group let us classify matrices $\hat{a}$, $\hat{b}$, $\hat{n}$ $\hat{k}$ in the following manner

A) $\hat{a} = \hat{I}$.

1) $\hat{k} = i(\alpha\hat{1} + \beta\hat{4}) + \gamma(\hat{2} - \hat{3})$;
2) $\hat{k} = i(\alpha\hat{1} + \beta\hat{4})$

a) $\hat{b}=0, \quad \hat{n}=n_1\hat{1}+n(\hat{2}+\varepsilon\hat{3})+n_4\hat{4} \quad (\varepsilon=\pm 1)$
b) $\hat{b}=b_1(\exp i\gamma)\hat{1}+æ b_2\hat{2}+\hat{3}b_3/æ+\hat{4}b_4\exp(-i\gamma)$.

B) $\hat{a}=\hat{1}, \quad \hat{k}=i\alpha\hat{1}+k_2\hat{2}+k_4\hat{4}$.
1) $\hat{b}=\alpha\hat{1}+\xi\hat{2}$, 2) $\hat{b}=\xi_1\hat{3}(\exp i\beta)+\xi\hat{4}$,
3) $\hat{b}=\hat{0} \quad (\xi_1,\xi=0,1)$.

C) $\hat{a}=\hat{1}+\hat{2}, \quad \hat{k}=i\alpha\hat{1}+k_2\hat{2}+k_4\hat{4}$.
1) $\hat{b}=\alpha\hat{1}+b\hat{2}, \quad \alpha b\neq 0$, 2) $\hat{b}=\hat{3}(\exp i\alpha)+b\hat{4}$,

D) $\hat{a}=\hat{b}=\hat{0}$.

The last version (D) was considered by Meshkov [3]. Solving the sets (13), (14), one has to consider next variants: I. det $\hat{B}\hat{Y}\neq 0$; II. det $\hat{B}\neq 0$, det $\hat{Y}=0$; III. det $\hat{B}=$ det $\hat{Y}=0$. For example, let det $\hat{B}\hat{Y}\neq 0$, then the set (13), (14) can be presented in the form:

$$2(B_3B_4T_1-Y_3Y_4T_2)=\mu[(\bar{b}_1B_4-\bar{b}_2B_3)/B+(\bar{a}_1Y_4-\bar{a}_2Y_3)/Y], \tag{17}$$

$$2(B_1B_2T_1-Y_1Y_2T_2)=\mu[(\bar{b}_4B_1-\bar{b}_3B_2)/B+(\bar{a}_4Y_1-\bar{a}_3Y_2)/Y], \tag{18}$$

$$\begin{aligned}&Y[(\bar{b}_1B_1+\bar{b}_3B_3)(B_2Y_3-B_4Y_1)(Y_4B_2-Y_2B_4)-\\&\quad-(\bar{b}_2B_2+\bar{b}_4B_4)(B_1Y_3-B_3Y_1)(Y_4B_2-Y_2B_4)]+\\&\quad+B[(\bar{a}_4Y_4+\bar{a}_2Y_2)(Y_3B_2-Y_1B_4)(Y_3B_1-Y_1B_3)-\\&\quad-(\bar{a}_1Y_1+\bar{a}_3Y_3)(Y_4B_1-Y_2B_3)(B_2Y_4-B_4Y_2)]=0;\end{aligned} \tag{19}$$

$$B(a_1\bar{Y}_2-a_2\bar{Y}_1+a_3\bar{Y}_4-a_4\bar{Y}_3)=Y(-\bar{b}_1B_2+\bar{b}_2B_1-\bar{b}_3B_4+\bar{b}_4B_3); \tag{20}$$

$$\begin{aligned}\Phi_0&=[2B_3B_4\mathscr{I}_1-\mu(\bar{b}_1B_4-\bar{b}_2B_3)]/B,\\ \Phi_1&=[-2(B_1B_4+B_2B_3)\mathscr{I}_1+\mu(\bar{b}_1B_2-\bar{b}_2B_1-\bar{b}_3B_4+\bar{b}_4B_3)]/2B,\\ \Phi_2&=[2B_1B_2\mathscr{I}_1+\mu(\bar{b}_3B_2-\bar{b}_4B_1)]/B,\\ \mathscr{I}_2&=-\mu(\bar{b}_1B_2-\bar{b}_2B_1+\bar{b}_3B_4-\bar{b}_4B_3)/2B.\end{aligned}$$

Here $B=\det\hat{B}$, $Y=\det\hat{Y}$, $\mathscr{P}_1=\mathscr{I}_1+\mathscr{I}_2$, $\mathscr{P}_2=\mathscr{I}_2-\mathscr{I}_1$, $T_1=\mathscr{I}_1/B$, $T_2=\bar{\mathscr{I}}_1/Y$, $\hat{B}=B_1\hat{1}+B_2\hat{2}+B_3\hat{3}+B_4\hat{4}$, $\hat{Y}=Y_1\hat{1}+Y_2\hat{2}+Y_3\hat{3}+Y_4\hat{4}$. The matrices $\hat{B}$, $\hat{Y}$ contain the arbitrary constants. To determine the constants one has to consider (17) and (20). The problem of finding these constants is not difficult but appropriate calculations are rather cumbersome. That is why we do not present them here. We give all nonequivalent matrices $\hat{B}$, $\hat{Y}$ and all electromagnetic fields Φ_0, Φ_1, Φ_2 for all variants in Appendix.

Appendix

I. $\det\hat{B}\det\hat{Y}\neq 0, \quad W=u^0u^1-u^2u^3$.

1) $\Phi_0=2iu^1u^3\theta(u)+\mu u^3\xi/W$,
$\Phi_1=-i(u^0u^1+u^2u^3)\theta(u)+\mu\xi(u^0-u^1)/2W$,
$\Phi_2=2iu^0u^2\theta(u)+\mu u^2\xi/W$,
$\hat{a}=\hat{I}, \quad \hat{b}=i\xi\hat{\sigma}_2, \quad \hat{k}=\hat{n}=0, \quad \xi=\pm 1$,
$\hat{L}=\square+2W\theta(u)\sigma_3+i\mu\xi(u^0+u^1)/W$.

2) $\Phi_0 = \mu(\alpha u^3 - \gamma u^1)/W, \quad \Phi_1 = \mu[\gamma(u^2 - u^3) - \alpha u^0 - \beta u^1]/2W,$
$\Phi_2 = \mu(\gamma u^0 + \beta u^2)/W,$
$\gamma^2 + \alpha\beta = -1, \quad \hat{a} = \hat{E}, \quad \hat{b} = \gamma\sigma_3 + \alpha\hat{2} + \beta\hat{3}, \quad \hat{k} = \hat{n} = 0,$
$\hat{L} = \Box + 2i\mu[\gamma(u^2 + u^3) - \alpha u^0 + \beta u^i]/W.$

3) $\Phi_0 = \mu[i\alpha c\gamma^2 - (\gamma^2 + c\bar{c})(\bar{c}(u^3)^2 + u^3(c\bar{c} - i\alpha))]/\gamma\alpha c\bar{c}(u^0 + i\beta),$
$\Phi_1 = \mu(\gamma^2 + c\bar{c})(\bar{c}u^3 + (c\bar{c} - i\alpha)/2)/\alpha\gamma c\bar{c},$
$\Phi_2 = -\mu(\gamma^2 + c\bar{c})(u^0 + i\beta)/\alpha\gamma c,$
$\hat{a} = \hat{1} + \hat{2}, \quad \hat{b} = i\gamma(\hat{1} + \hat{2})/c,$
$\hat{k} = i\beta(\hat{1} + \hat{2}) + (c - i\alpha/\bar{c})\hat{4}, \quad \hat{n} = -\alpha\gamma/c^2(\hat{4} - \hat{3}),$
$\hat{L} = \Box + i\mu(i\alpha - c\bar{c})(\gamma^2 + c\bar{c})\sigma_3/2\alpha\gamma c\bar{c}.$

II. $\det\hat{B}\det\hat{Y} = 0.$

4) $\Phi_0 = -[\mu(1 + c\bar{c})u^3 + \mu\bar{c}\alpha - 2i(cu^3 + \alpha)u^3\theta(u)]/\alpha(u^0 + i\beta),$
$2\alpha\Phi_1 = \mu(1 + c\bar{c}) - 2i(2cu^3 + \alpha)\theta(u),$
$\Phi_2 = 2ic(u^0 + i\beta)\theta(u)/\alpha,$
$\hat{a} = \hat{1}, \quad \hat{b} = c\hat{1}, \quad \hat{k} = i\beta\hat{1}, \quad \hat{n} = ic\beta\hat{1} + \alpha\hat{3},$
$\hat{L} = \Box + 2\theta + i\mu(1 + c\bar{c})/\alpha.$

III. $\hat{a} = \hat{b} = 0.$

5) $\Phi_0 = 0, \quad \hat{k} = \hat{1}, \quad \hat{n} = n\hat{2}, \quad \hat{L} = \Box + 2i(\Phi_1\hat{1} + \Phi_1\hat{4}).$

In these formulas the real constants are denoted by small Greek letters, the complex ones by small Latin letters. We do not consider here the case $R_{ijkl} \neq 0$. Note that some of the presented fields depend on m. Apparently these variants can be used to study the integration problem of the Dirac–Maxwell equation.

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