PRAMANA
— journal of
physics



Vol. 41, No. 6,
December 1993
pp. 493–502

# Energy levels of a two-dimensional anharmonic oscillator: Hill determinant approach

M R M WITWIT
Department of Applied Mathematics, University of Hull, Hull, HU6 7RX, UK

MS received 28 June 1993; revised 8 September 1993

**Abstract.** The energy levels of the Schrödinger equation for the potential $x^2 + y^2 + \lambda[a_{xx}x^4 + 2a_{xy}x^2y^2 + a_{yy}y^4]$ are calculated using Hill determinant approach for several eigenstates and over a wide range of values of perturbation parameters. The obtained numerical results agree with those previously reported by other methods.

**Keywords.** Schrödinger equations; Hill determinant; numerical results.

**PACS No.** 03·65

## 1. Introduction

Harmonic oscillator models have played an important role in the evolution of a number of areas of physics ranging from vibrations of crystal lattices to black body radiation to elasticity to acoustic, to molecular dynamics, etc.

This paper aims to calculate the energy levels of the two-dimensional oscillator which is expressed by the potential

$$V(x, y) = x^2 + y^2 + \lambda[a_{xx}x^4 + 2a_{xy}x^2y^2 + a_{yy}y^4]. \tag{1}$$

The two-dimensional anharmonic oscillator $V(x, y)$ has received a lot of attention recently as it is related to several interesting physical problems. Many techniques have been used to obtain the energy eigenvalues [1–8].

In this paper, some extended numerical calculations are, presented, using the Hill determinant approach (iterative technique) to calculate the energy levels of the Schrödinger equation for the potential (1) over a wide range of values of the perturbation parameters and for various eigenstates. The convergence of the energy depends upon the choice of an adjustable parameter $\beta$. However the convergence rate is relatively good, with better choice of $\beta$ as will see later.

In rectangular coordinates the Schrödinger equation for the potential $V(x, y)$ can be written as:

$$\left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + V(x, y)\right]\Psi_{nx,ny}(x, y) = E\Psi_{nx,ny}(x, y). \tag{2}$$

The energy $E_{nx,ny}$ and wavefunction $\Psi_{nx,ny}(x, y)$ corresponding to the Schrödinger equation (2) when the perturbation parameter $\lambda = 0$ can be expressed as:

$$E_{nx,ny} = (2nx + 1) + (2ny + 1) \equiv 2n + 2 \quad n_x, n_y = 0, 1, 2\ldots \tag{3}$$

$$\Psi_{nx,ny}(x,y) = \phi_{nx}(x)\phi_{ny}(y), \tag{4}$$

$$\phi_{nx}(x) = \left[\frac{1}{2^{nx}nx!}(\alpha/\pi)^{\frac{1}{4}}\right] \quad H_{nx}(\sqrt{\alpha}x)\exp -(\alpha/2)x^2. \tag{5}$$

As indicated earlier [7], eq (2) has a circular symmetry when the relationship $a_{xx} = a_{yy} = a_{xy} = 1$ holds for the potential (2). The energy levels are then most appropriately characterized by the quantum numbers $(n_r, M)$ rather than $(n_x, n_y)$. The radial part of the eigenvalue equation (2) can be expressed as:

$$\left[-\frac{d^2}{dr^2} + \left[M^2 - \frac{1}{4}\right]r^{-2} + r^2 + \lambda r^4\right]\Psi(r) = E\Psi(r). \tag{6}$$

The unperturbative energy in polar coordinates can be expressed as:

$$E_{nr,M} = 4n_r + 2|M| + 2 \equiv E_{nx,ny} \tag{7}$$

where $n_r = 0, 1, 2, 3, \ldots$ and $M = 0, \mp 1, \mp 2, \ldots$ are the orbital quantum number and the magnetic quantum number.

In each state the energy levels depend on the pair quantum numbers $(n_x, n_y)$. The degeneracy degree of $n$th level is equal to the number of ways in which $n$ can be divided into the sum of two positive integral (or zero) numbers; which is $n + 1$.

In (2) we have supposed that, to each eigenvalue $E_{nx,ny}$ there corresponds just one eigenfunction $\Psi_{nx,ny}(x, y)$. In the degenerate case several eigenfunctions correspond to the same eigenvalue. We notice that, as a result of the perturbation, an originally degenerate energy level ceases in general to be degenerate, because the perturbation removes the degeneracy. For example the states (1, 1; even-0, 2; even-2, 0; odd) start off to degenerate at $\lambda = 0$, but split up into three separate levels respectively as $\lambda$ increases.

## 2. Hill determinant approach and its implementation to the two-dimensional oscillator and its recurrence relations

The Hill determinant approach is a non-perturbative technique and can handle numerical calculations of the eigenvalue problem in one-dimensional case for various types of potentials [10, 11, 12].

The polynomial potential given by (1) is non-separable in cartesian coordinates, but shows a high symmetry. Due to this symmetry it is possible to cut down the amount of computation needed in the program. The more general anisotropic case can also be treated by our technique.

We stress that the Hill determinant approach has been found comparable to the inner product method when applied to the same perturbation in multidimensional systems.

To find the recurrence relations which allow us to calculate the eigenvalues, we introduce the following wavefunction:

$$\Psi_{nx,ny}(x,y) = \exp\left[-\frac{\beta}{2}(x^2 + y^2)\right]\sum_{L,I} F(L,I)(x^L y^I). \tag{8}$$

By substituting $\Psi_{nx,ny}(x, y)$ in (2) and after some algebra, we get the following

recurrence relation,

$$F(L,I) = W(L,I)[2\beta(L+I+1) - E]^{-1}, \tag{9}$$

where

$$W(L,I) = (L+2)(L+1)F(L+2,I) + (I+2)(I+1)F(L,I+2) + (\beta^2 - 1)[F(L-2,I) + F(L,I-2)] - \lambda[a_{xx}F(L-4,I) + a_{yy}F(L,I-4) + 2a_{xy}F(L-2,I-2)]. \tag{10}$$

For calculations, we set the initial coefficient $F(L_0, I_0) = 1$. All the $S(L,I)$ with $(L,I) \neq (L_0, I_0)$ are treated according to (9) for some trial values of $\beta$ and $E$. From experience it is seen that the best initial energy to start the calculation should be a little higher than the required eigenvalue [13]. The value of $\beta$ increases as the perturbation parameter $\lambda$ increases. At this stage, naturally, the question that arises is whether $\beta$ can give the best convergence eigenvalue. In the present calculations, the value of $\beta$ which gave the best converged eigenvalue was taken. Table 1 shows the rate of convergence by this technique for some eigenstates for $\lambda = 0{\cdot}1$, 10, 100, $10^5$ in the special case, wherein the potential has circular symmetry. It is known from experience, that the rate of the convergence decreases as values of the perturbation parameter $\lambda$ and eigenstate number $(n_x, n_y)$ increase.

The standard technique for evaluation of the energy is based on (9), that for the special case $L = L_0$, $I = I_0$, the coefficient on the left-hand side becomes $F(L_0, I_0) = 1$, and this initial condition is imposed on the algorithm. After this adjustment a revised $E$ estimate is calculated using the assignment statements

$$E_e = 2\beta(L_0 + I_0 + 1) - W(L_0, I_0). \tag{11}$$

$$E = RE_e + (1-R)E. \tag{12}$$

The value of the relaxation parameter $R$ can be decreased or increased to help in improving and stabilizing a convergence of the eigenvalue to a reasonable accuracy. The parity index $P_{xy}$ (for interchange coordinates $x \leftrightarrow y$) is used for the potential $V(x,y)$, in the case of symmetry interaction which means $a_{IJ} = a_{JI}$. The relation between the coefficients hold (for interchange $x \leftrightarrow y$).

$$F(L,I) = P_{xy}F(I,L). \tag{13}$$

For even eigenstates $P_{xy} = 1$ and for odd eigenstates $P_{xy} = -1$ and $\beta$ is varied to give the best possible convergence of the estimated energy.

## 3. Results and discussion

The Hill determinant approach has been applied in this paper to the Schrödinger equation with a perturbed potential in a two-dimensional system. It is found very effective to treat symmetry perturbation i.e. $a_{IJ} = a_{JI}$ and unsymmetry perturbation i.e. $a_{IJ} \neq a_{JI}$ for several eigenstates and various values of perturbation parameters.

The adjustable parameter $\beta$ plays an important role in the convergence aspects of the present calculations. The $\beta$ values for the dimensional oscillator $V(x,y)$ calculation are in the range $1{\cdot}3 \leqslant \beta \leqslant 350$. The general consideration that governs our choice is that, as $\lambda$ increases, $\beta$ also increases, for example $\beta = 1{\cdot}3$ at $\lambda = 0{\cdot}01$ and $\beta = 350$ at $\lambda = 10^6$.

**Table 1.** Convergence for some eigenvalues for $(n_x, n_y; \pi)$ of the two dimensional oscillator for the case $a_{xx} = a_{yy} = a_{zz} = 1$, for $\lambda = 0{\cdot}1, 10, 100, 10^5$ with various values of the adjustable parameter $\beta$. The empty spaces mean the corresponding eigenvalues cannot be obtained with these values of $\beta$.

| $\lambda$ | $(0,0;+)$ | $\beta$ | $(1,0;\mp)$ | $\beta$ | $(1,1;+)=(2,0;-)$ | $\beta$ |
|---|---|---|---|---|---|---|
| | | 0·5 | | 1 | | 1 |
| | 2·16 | 1 | 4·4776003 | 1·5 | 6·9083321 | 1·6 |
| 0·1 | 2·1685972 | 1·4 | 4·47760036076 | 1·7 | 6·908332111231871 | 2·0 |
| | 2·1685972112693 | 1·8 | 4·47760036076821 | 2·0 | 6·908332111231871 | 2·2 |
| | 2·168597211269291 | 2·5 | 4·47760036076821 | 2·5 | 6·908332111231871 | 2·5 |
| | | 3 | | 4 | | 5 |
| | 5·3493 | 6 | 12·138224 | 7 | 19·936 | 7 |
| 10 | 5·349352819406 | 8 | 12·1382247388 | 8 | 19·936900374 | 8 |
| | 5·34935281940641 | 9 | 12·13822473889884 | 9 | 19·9369003740113 | 9 |
| | 5·34935281940641 | 10 | 12·13822473889884 | 10 | 19·9369003740113 | 10 |
| | | 8 | | 8 | | 10 |
| | 11·023 | 12 | 25·27998 | 14 | 41·76874 | 15 |
| | 11·0235979 | 15 | 25·2799851433 | 17 | 41·76874434397 | 18 |
| 100 | 11·0235979287809 | 20 | 25·27998514334519 | 20 | 41·7687443439749 | 20 |
| | 11·0235979287809 | 24 | 25·27998514334519 | 24 | 41·7687443439749 | 24 |
| | | 80 | | 90 | | 100 |
| | 108·85135 | 140 | 250·402 | 140 | 414·4377 | 150 |
| $10^5$ | 108·8513595772 | 180 | 250·4021140684 | 180 | 414·4377597604 | 180 |
| | 108·851359577201 | 200 | 250·402114068453 | 200 | 414·437759760461 | 200 |
| | 108·851359577201 | 220 | 250·402114068453 | 220 | 414·437759760461 | 220 |

**Table 2.** Eenrgy values of $(n_x, n_y; \pi)$, for the potential $V(x, y)$ for several sets of perturbation parameters. The parity label $\pi = +, -$; even, odd for the $x \leftrightarrow y$ interchange symmetry.

| $a_{xx} = a_{yy} = 0, a_{xy} = \frac{1}{2}$ | | | | |
|---|---|---|---|---|
| $\lambda$ | $(0,0;+)$ | $(1,0;\pm)$ | $(1,1;+)$ | $(2,0;-)$ |
| 0·1 | 2·02413832141573 | 4·0708639128944 | 6·2082907985657 | 6·0716400174184 |
| 0·2 | 2·04679571883258 | 4·1350822342067 | 6·3912911417583 | 6·1377408133565 |
| 0·3 | 2·068242130731 | 4·1942935736199 | 6·5564773059808 | 6·1995318631918 |
| 0·4 | 2·088667659069 | 4·2495313630102 | 6·7081422885770 | 6·2578183515335 |
| 0·5 | 2·108213779698 | 4·301504505315 | 6·8490530489122 | 6·313168654731 |
| 0·6 | 2·126989929539 | 4·35072736225 | 6·9811280772305 | 6·36600479493 |
| 0·7 | 2·1450832429 | 4·3975886228 | 7·1057672375444 | 6·4166518870 |
| 0·8 | 2·1625646859 | 4·4423912545 | 7·2240314806597 | 6·465367562 |
| 0·9 | 2·1794931153 | 4·485377268 | 7·3367489236663 | 6·512360604 |
| 1·0 | 2·195918086 | 4·526743874 | 7·444581361569 | 6·557803326 |
| 2·0 | 2·3395662 | 4·87770957 | 8·3394493229 | 6·9506339 |
| 3·0 | 2·4583769 | 5·1579605 | 9·034854244 | 7·271682 |
| 4·0 | 2·5616266 | 5·396806 | 9·6175878 | 7·549157 |
| 5·0 | 2·6539097 | 5·60757 | 10·1256931 | 7·79644 |
| 6·0 | 2·7379188 | 5·79769 | 10·5798578 | 8·0212 |
| 7·0 | 2·8153967 | 5·97184 | 10·0992791 | 8·2283 |
| 8·0 | 2·8875504 | 6·13341 | 11·372950 | 8·4214 |
| 9·0 | 2·955257 | 6·28596 | 11·726290 | 8·6019 |
| 10·0 | 3·019178 | 6·42591 | 12·057188 | 8·773 |
| 15·0 | 3·29685 | 7·03842 | 13·469681 | 9·515 |
| 20·0 | 3·52692 | 7·54125 | 14·616918 | 10·131 |
| 50·0 | 4·47652 | 9·60194 | 19·21721 | 12·685 |
| 100 | 5·46097 | 12·2666 | 23·86291 | 15·349 |
| 1000 | 11·23243 | 24·2562 | 50·39571 | 31·13 |
| 5000 | 19·03854 | 41·2135 | 85·85665 | 52·61 |
| 10000 | 23·94599 | 51·6251 | 108·0969 | 66·13 |
| $10^5$ | 51·4718 | 110·5191 | 232·6638 | 142·2 |

| | $a_{xx}=a_{yy}=1,\ a_{xy}=-1$ | | $a_{xx}=\frac{1}{2},\ a_{yy}=\frac{1}{4},\ a_{xy}=\frac{1}{8}$ | |
|---|---|---|---|---|
| $\lambda$ | $(0,0;+)$ | $(1,0;\pm)$ | $(1,0;-)$ | $(1,1;+)$ |
| 0·01 | 2·00985521060372 | 4·0292889726409 | 4·02220907755723 | 6·033322273821 |
| 0·1 | 2·08866765907315 | 4·2495313630310 | 4·2014184255855 | 6·302730175585 |
| 0·5 | 2·339566211 | 4·87770957 | 4·776661317376 | 7·170817128601 |
| 1 | 2·56162658 | 5·3968039 | 5·28444637458 | 7·9385946295 |
| 10 | 4·21175722 | 9·02688 | 8·99050466 | 13·54179268 |
| 100 | 8·36832652 | 17·984 | 18·139463 | 27·355525 |
| 1000 | 17·68691 | 37·96 | 38·4951 | 58·0717 |
| 2000 | 22·2399 | 47·74 | 48·4255 | 73·0545 |
| 10000 | 37·945 | 81·48 | 82·6619 | 124·7086 |

| | $a_{xx}=1,\ a_{yy}=\frac{1}{2},\ a_{xy}=\frac{1}{4}$ | | $a_{xx}=\frac{1}{3},\ a_{yy}=\frac{1}{4},\ a_{xy}=\frac{1}{2}$ | |
|---|---|---|---|---|
| $\lambda$ | $(0,0;+)$ | $(1,0;-)$ | $(0,0;+)$ | $(1,0;-)$ |
| 0·01 | 2·01231228718489 | 4·0438677204024 | 2·0068218851034 | 4·021621108365 |
| 1 | 2·6863516426 | 6·02495513951 | 2·4550321766714 | 5·281706465822 |
| 5 | 3·8255941189 | 8·9905464633 | 3·32198865267 | 7·47632663472 |
| 10 | 4·619200437 | 10·991656058 | 3·95089995457 | 9·01371371691 |
| 50 | 7·50300538 | 18·13946345 | 6·28890732395 | 14·6138003405 |
| 100 | 9·3555061 | 22·69326466 | 7·80745103125 | 18·2150199295 |
| 500 | 15·809344 | 38·4950971 | 13·12621353 | 30·767112789 |
| 1000 | 19·87269 | 48·425489 | 16·48332214 | 38·671655693 |
| 10000 | 42·6818 | 104·112662 | 35·35459 | 83·04905 |
| $10^5$ | 91·8949 | 224·20318 | 76·09585 | 178·7999 |

| | $a_{xx}=\frac{1}{3},\ a_{yy}=\frac{1}{6},\ a_{xy}=\frac{1}{5}$ | | $a_{xx}=0,\ a_{yy}=1,\ a_{xy}=\frac{1}{2}$ | |
|---|---|---|---|---|
| $\lambda$ | $(0,1;+)$ | $(1,0;-)$ | $(0,0;+)$ | $(1,1;+)$ |
| 0·01 | 4·0116729015606 | 4·016594932356 | 2·00982882574815 | 6·058318025551 |
| 0·1 | 4·1106369197386 | 4·154229422204 | 2·08694982082748 | 6·485730943303 |
| 0·5 | 4·466738393841 | 4·622448523457 | 2·32310658866 | 7·65887777919 |
| 1 | 4·81001774999 | 5·053036645166 | 2·52658354 | 8·6065492 |
| 10 | 7·59082621315 | 8·335444773 | 4·00772 | 15·05018 |
| 100 | 14·8799478 | 16·63230785 | 7·7615 | 30·53609 |
| 1000 | 31·3501683 | 35·2005676 | 16·26 | 64·8681 |
| 5000 | 53·387759 | 59·9954 | 27·67 | 110·636 |
| 10000 | 67·2109 | 75·5418 | 34·83 | 139·323 |
| 100000 | 144·648 | 162·612 | 74·9 | 299·96 |

| | $a_{xx}=\frac{1}{3},\ a_{yy}=\frac{1}{4},\ a_{xy}=\frac{1}{2}$ | $a_{xx}=\frac{1}{4},\ a_{yy}=\frac{1}{5},\ a_{xy}=\frac{1}{6}$ | $a_{xx}=a_{yy}=\frac{1}{3},\ a_{xy}=\frac{1}{4}$ | |
|---|---|---|---|---|
| $\lambda$ | $(1,1;+)$ | $(1,1;+)$ | $(0,2;+)$ | $(2,0;-)$ |
| 0·01 | 6·0436934891649 | 6·0241668378937 | 6·043135666616 | 6·038232473314 |
| 1 | 8·3665961639843 | 7·5602656974176 | 8·38684555658 | 8·164167323602 |
| 5 | 12·1540605363882 | 10·4238563537005 | 12·2283227870 | 11·733936775982 |
| 10 | 14·75829350949 | 12·471386219668 | 14·8663230227 | 14·207348368038 |
| 50 | 24·149078364 | 20·0155548073 | 24·3660855 | 23·1626619844 |
| 100 | 30·158605748 | 24·893689976 | 30·4406285 | 28·9045928065 |
| 500 | 51·0558236 | 41·9417026 | 51·555132 | 48·8897072 |
| 1000 | 64·201194 | 52·690847 | 64·834505 | 61·4667308 |
| 10000 | 137·95683 | 113·0797 | 139·3334 | 132·05022 |
| 100000 | 297·0515 | 243·418 | 300·0232 | 284·3189 |

| | $a_{xx}=a_{yy}=a_{xy}=1$ | | | $a_{xx}=a_{yy}=1,\ a_{xy}=0$ |
|---|---|---|---|---|
| $\lambda$ | $(0,0;+)$ | $(0,1;\mp)$ | $(1,1;+)=(2,0;-)$ | $(1,1;+)$ |
| 0·01 | 2·0195708085219 | 4·0581733885884 | 6·115306125265 | 6·073056060903 |
| 1 | 2·952050091966 | 6·46290599986 | 10·390627295503 | 9·297625408424 |
| 5 | 4·3810888395 | 9·87059482807 | 16·146296004255 | 14·026958371634 |
| 10 | 5·3493528194 | 12·13822473884 | 19·936900374011 | 17·198006909616 |
| 50 | 8·814283232 | 20·1770296484 | 33·304412267221 | 28·48341399182 |
| 100 | 11·023597928 | 25·2798514335 | 41·768744343975 | 35·66038543191 |
| 500 | 18·69290316 | 42·9558103393 | 71·054463932988 | 60·5430105116 |
| 1000 | 23·51338918 | 54·0548557958 | 89·943343403375 | 76·173669187 |
| 10000 | 50·54804495 | 116·267380968 | 192·420561201906 | 163·806633902 |
| 100000 | 108·85135957 | 250·402114065 | 414·437759760460 | 352·769246001 |
| $10^6$ | 234·48942203 | 539·433981018 | 892·823487959 | 759·952638 |

| | $a_{xx}=a_{yy}=1,\ a_{xy}=0$ | | $a_{xx}=1\ \ a_{yy}=\frac{1}{2},\ a_{xy}=1$ | |
|---|---|---|---|---|
| $\lambda$ | $(0,0;+)$ | $(0,1;\mp)$ | $(0,1;+)$ | $(1,0;-)$ |
| 0·01 | 2·0147473441627 | 4·0438989765947 | 4·040310310588 | 4·054616746577 |
| 1 | 2·7847032830606 | 6·0411643457423 | 5·929093989647 | 6·361312813522 |
| 5 | 4·0366812987306 | 9·0318198351823 | 8·8068361776 | 9·66976839748 |
| 10 | 4·898348144236 | 11·0481775269262 | 10·7552685288 | 11·87757748425 |
| 50 | 8·007985536555 | 18·245699764189 | 17·7255829568 | 19·7159195444 |
| 100 | 9·998835090275 | 22·829610261090 | 22·1694645877 | 24·6950247031 |
| 500 | 16·923285258162 | 38·733147884922 | 35·5949254503 | 41·9482797882 |
| 1000 | 21·279577422656 | 48·72662217071 | 47·2902582298 | 52·7836179925 |
| 10000 | 45·723217740545 | 104·76492582355 | 101·6637536694 | 113·5233045707 |
| 100000 | 98·450895516845 | 225·61007058467 | 218·9257185534 | 244·4877262122 |
| $10^6$ | 212·07965847 | 486·01614822 | 471·611371845 | 526·69067889 |

The results are in good agreement with available literature. For accurate results, the best values of $\beta$, have been chosen, which were obtained by testing many parameter values until the best convergence was obtained.

Some important consequences of our investigation are as follows:

1. The Hill determinant approach can be used to calculate several eigenstates with excellent accuracy for the two-dimensional oscillator for several perturbation parameters. Table 2 covers a wide range of $\lambda$ values ($10^{-2} \leqslant \lambda \leqslant 10^{6}$) and various cases wherein the potential has interchange symmetry. The $\lambda$ value obtained by us is twice the $\lambda$ value obtained by Nasit and Demiralp [1] and Hioe *et al* [3] and their Hamiltonian differs by a factor 1/2.

Several external checks were devised to verify the correctness of the energy values. For example using the inner product technique, we obtained agreement between some of the present results and those obtained by inner product [7]. For the two special cases $a_{xx} = a_{yy} = a_{xy} = 1$ and $a_{xx} = a_{yy} = 1$, $a_{xy} = 0$, the potential (1) has circular symmetry, and reduces to the two independent oscillators respectively. We have checked the energies obtained by the present technique against results obtained by the hypervirial method [7].

2. For symmetric perturbation i.e. $a_{IJ} = a_{JI}$, it is seen from our calculations that the energy levels characterized by eigenstates $(n_x, n_y,)$ with $n_x$ and $n_y$ having different parity, i.e. (odd, even) or (even, odd) remain doubly degenerated and unsplit as $\lambda$ is varied from zero value. This means that the perturbation does not break the degeneracy of the perturbed system. The energy levels which show such behaviour in our calculations are $(E_{10}, E_{01})$. Unsymmetry perturbation i.e. $a_{IJ} \neq a_{JI}$ shows interesting splitting effects as $a_{IJ}$ and $a_{JI}$ vary. In this case the degeneracy is broken by the perturbation and this behaviour is clear from our calculations in table 2.

Also there is a crossing between the energy levels for the two eigenstates $E_{11}^{+}$ and $E_{20}^{-}$ at $a_{xx} = a_{yy} = a_{xy} = 1$; (circular symmetry), but when this symmetry is removed, the doubly degenerate levels split into different levels, and this confirmed by our results in table 2.

3. The phenomenon of bogus convergence is avoided by computing the energy eigenvalues for different values of the adjustable parameter $\beta$. To select the correct converge energy we require stability of the results with respect to the small variation of $\beta$ at a given value of the perturbation parameter $\lambda$.

We have also used Aitken's transformation [14] to improve the convergence of the calculations and it has helped in improving our results.

**Acknowledgement**

The author is grateful to the referee for valuable comments and suggestions.

**References**

[1] Ari Nasit and Metin Demiralp, *J. Math. Phys.* **26**, 1179 (1985)
[2] R Blankenbecler, T DE Grand and R L Sugar, *Phys. Rev.* **D21**, 1055 (1980)
[3] F T Hioe, Don MacMkllen and E W Montroll, *Phys. Rep.* **43** 305 (1978)
[4] F M Fernandez, A M Meson and E A Castro, *Phys. Lett.* **A112**, 107 (1985)
[5] J P Killingbeck and M N Jones, *J. Phys.* **A19**, 705 (1986)
[6] J P Killingbeck, *Microcomputer algorithms* (Bristol, Adam Hillger, 1991)
[7] M R M Witwit, *J. Phys.* **A24**, 4035 (1991)

[8] F Arias de Saavedra and E Buendia, *J. Phys.* **A24** L1209 (1991)
[9] M R M Witwit, *J. Phys.* **A24**, 3053 (1991)
[10] S N Biswas, K Datta, R P Saxena, P K Srivastava and V S Varma, *Phys. Rev.* **D4**, 3617 (1971)
[11] S N Biswas, K Datta, R P Saxena, P K Srivastava and V S Varma, *J. Math. Phys.* **14**, 1190 (1973)
[12] K Banerjee, *Proc. R. Soc. London* **A364**, 265 (1978)
[13] M R M Witwit, *J. Math. Phys.* **33**, 4196 (1992)
[14] A C Aitken, *Proc. Edinburgh Math. Soc.* **3** 56 (1932)