PRAMANA
— journal of
physics

Vol. 41, No. 6,
December 1993
pp. 483–491

Uniqueness of the solutions of multichannel scattering equation in three particle system

SUPROKASH MUKHERJEE
Theoretical Physics Group, Saha Institute of Nuclear Physics, Bidhannagar, Calcutta 700064, India

MS received 19 January 1993; revised 23 August 1993

Abstract. A new approach is made to investigate the old problem of non-uniqueness of the solution of Lippmann–Schwinger equation of three particle system and it is found that even the necessary condition for the existence of the problem is not satisfied.

Keywords. Lippman–Schwinger equation; multichannel scattering.

PACS Nos 03·65; 25·10

1. Introduction

The solutions of the inhomogeneous Lippmann–Schwinger (LS) equations for three particle system are considered to be non-unique [1]. The scattering states $|\psi_\alpha^\pm(E_\alpha)\rangle$ in α-channel at energy E_α, satisfy one inhomogeneous LS-equation and several homogeneous LS-equations, so that the solution $|\psi_\alpha^\pm(E_\alpha)\rangle$ of the inhomogeneous LS-equation have arbitrary admixture of the similar other states $|\psi_\beta^+(E_\alpha)\rangle$ with $\beta \neq \alpha$. That is, $|\psi_\alpha^\pm(E_\alpha)\rangle$ as well as $|\Phi_\beta^\pm(E_\alpha)\rangle = |\psi_\alpha^\pm(E_\alpha)\rangle + \sum_{\beta\neq\alpha} C_\beta|\psi_\beta^\pm(E_\alpha)\rangle$ with arbitrary coefficient C_β, are solutions of same inhomogeneous LS-integral equation, as straightforward algebra would seem to indicate. Such analysis, of course, tacitly assumes without proof that the homogeneous LS-equation have non-trivial solutions and that these are considered to be same functions as those obtained by solving appropriate inhomogeneous LS-equations, as Lippmann's identity seem to indicate. However, in the Fredholm theory of simultaneous integral equations one obtains (Colton and Kress [2]) an extra condition of consistency, called solvability condition for simultaneous existence of inhomogeneous and homogeneous integral equations for the same Kernel. In the light of such analysis in Fredholm theory, we reexamine the question about the condition of coexistence of homogeneous and inhomogeneous LS-equations and do find that this requires that the transition matrix of the rearrangement scattering be identically zero, at any arbitrary energy. This striking new result, therefore, rules out the coexistence of the homogeneous and inhomogeneous LS-equations vis-a-vis the said problem of the non-uniqueness of the solutions of LS-equations. These are discussed in § 2 along with its consequences. It is argued that this result is an indirect proof that either Lippmann's identity is wrong, as it is responsible for the appearance of the homogeneous LS-equations, or that the homogeneous LS-equation if exist, can admit only trivial (null) solutions, so that the said non-uniqueness problem does not arise in any case. This conclusion is similar to that obtained earlier [3] which showed that the Lippmann's identity is wrong if LS-equations are regarded as valid in the distribution theoretic sense.

2. Theory

In the Fredholm theory of integral equations it can be proved [2,4] that the solution of the inhomogeneous integral equation, when it exists, is non-unique if the corresponding homogeneous integral equation has non-trivial solutions provided some conditions of consistency are satisfied. Corresponding to the Fredholm Kernel $K(x, y)$, let the homogeneous equation has non-trivial solutions

$$\phi_m(x) = \int K(x, y)\phi_m(y)\mathrm{d}y \tag{1}$$

for $m = 1, 2, \ldots, n$ then the claim is that the inhomogeneous integral equation for the same Kernel

$$\phi(x) = f(x) + \int K(x, y)\phi(y)\mathrm{d}y \tag{2}$$

for a square integrable inhomogeneous term $f(x)$, has either no solution, or when solvable, the L_2-solution of it is non-unique, being of the form

$$\phi(x) = \tilde{\phi}(x) + \sum_{m=1}^{n} C_m\phi_m(x) \tag{3}$$

where $\tilde{\phi}(x)$ is a particular solution of (2), C_m's are arbitrary constants, $\phi_1(x)$, $\phi_2(x), \ldots, \phi_m(x)$ are linearly independent solutions homogeneous equation of (1) (see corollary 1.19 [2]. In addition to this, in order that (1) is compatible with (2), a consistency condition, called solvability condition (see Th. 1.29 of [2]) is required to be satisfied, which demands that the inhomogeneous term $f(x)$ of (2) be orthogonal to all the solutions of the adjoint homogeneous equation, which is

$$\langle \psi_m | f \rangle = \int \psi_m(x) f(x)\mathrm{d}x = 0 \tag{4}$$

where $\psi_m(x)$ satisfies

$$\psi_m(x) = \int \mathrm{d}y \psi_m(y) K(y, x) \tag{5}$$

for $m = 1, 2, \ldots, n$. It can further be shown that in Fredholm theory, the condition (4) is a necessary and sufficient one that the inhomogeneous equation (2) has nontrivial solutions wherever the homogeneous equation (1), coexists with it, (2), nontrivially. In a multichannel scattering theory, the scattering state at a given energy satisfies the homogeneous and inhomogeneous Lippmann–Schwinger equation simultaneously and one concludes Glöckle [1] that the solution of the LS-equation is non-unique, perhaps in analogy with what is contained in (3) above. However, in the literature of multichannel scattering there is no mention of any solvability conditions corresponding to (4), which happens to be a necessary and sufficient condition for the coexistence of homogeneous and inhomogeneous equations (1) and (2). Although this result of L_2-theory of Fredholm integral equation cannot be applied to the LS-equation for $|\psi_\alpha^+(E_\alpha)\rangle$ (as $|\phi_\alpha(E_\alpha)\rangle$ and $V^\alpha|\psi_\alpha^\pm(E_\alpha)\rangle$ are not L_2-objects), one can still find a necessary condition for coexistence homogeneous and inhomogeneous equations of scattering.

The scattering states $|\psi_\alpha^\pm(E_\alpha)\rangle$ for α-channel, at a given energy E_α, are given by the following Lippmann–Schwinger equations: (we use the notation of Glöckle [1], mostly, with minor changes)

$$|\psi_\alpha^\pm(E_\alpha)\rangle = |\phi_\alpha(E_\alpha)\rangle + G_\alpha^\pm(E_\alpha)\, V^\alpha|\psi_\alpha^\pm(E_\alpha)\rangle \tag{6}$$

The superscripts + (or −) refer to solutions with outgoing (or incoming) wave boundary condition. This equation is normally derived by using the following equations:

$$|\psi_\alpha(E_\alpha + i\eta)\rangle = i\eta G_\alpha(E_\alpha + i\eta)|\phi_\alpha(E_\alpha)\rangle \tag{7}$$

$$|\psi_\alpha^\pm(E_\alpha)\rangle = \lim_{\eta\to 0^+} |\psi_\alpha(E_\alpha \pm i\eta)\rangle \tag{8}$$

$$G_\alpha(Z) = (Z - H_\alpha)^{-1}, \quad G_\alpha^\pm(E_\alpha) = \lim_{\eta\to 0^+} G_\alpha(E_\alpha \pm i\eta - H_\alpha)^{-1} \tag{9}$$

$$G(Z) = (Z - H)^{-1} = G_\alpha(Z) + G_\alpha(Z)\, V^\alpha G(Z) \tag{10}$$

The total Hamiltonian is $H = H_0 + V = H_\alpha + V^\alpha$; H_0 and $V = V_\alpha + V^\alpha$ being, respectively, the total kinetic and potential energy operators; the α-channel Hamiltonian $H_\alpha = H_0 + V_\alpha$ has its eigenfunctions $|\phi_\alpha(E_\alpha)\rangle$ with energy E_α. If one uses the Lippmann's identity given by:

$$\lim_{\eta\to 0^+} i\eta G_\beta(E_\alpha + i\eta)|\phi_\alpha(E_\alpha)\rangle = \delta_{\alpha\beta}|\phi_\alpha(E_\alpha)\rangle. \tag{11}$$

then, from (7) expressing $G(Z)$ in terms of $G_\beta(Z)$ through (10), we have the homogeneous equation of scattering, from (7), (8), (10) and (11):

$$|\psi_\alpha(E_\alpha + i\varepsilon)\rangle = i\eta G_\beta(E_\alpha + i\eta)|\phi_\alpha(E_\alpha)\rangle + G_\beta(E_\alpha + i\eta)\, V^\beta|\psi_\alpha(E_\alpha + i\eta)\rangle \tag{12}$$

$$|\psi_\alpha^\pm(E_\alpha)\rangle = G_\beta^\pm(E_\alpha)\, V^\beta|\psi_\alpha^\pm(E_\alpha)\rangle \tag{13}$$

where $\beta \neq \alpha$. Equation (13) gives the homogeneous equations of the scattering state $|\psi_\alpha^\pm(E_\alpha)\rangle$ (see eqn (3.51) of Glockle [1]), which yields a number of homogeneous equations, for all β other than $\beta = \alpha$. Again, for channel β we, similarly, obtain

$$|\psi_\beta^\pm(E_\alpha)\rangle = |\phi_\beta(E_\alpha)\rangle + G_\beta^\pm(E_\alpha)\, V^\beta|\psi_\beta^+(E_\alpha)\rangle \tag{14}$$

$$|\psi_\beta^\pm(E_\alpha)\rangle = G_\alpha^\pm(E_\alpha)\, V^\alpha|\psi_\beta^\pm(E_\alpha)\rangle \tag{15}$$

where $\phi_\beta(E_\alpha)$ is an eigenfunction of $H_\beta = H_0 + V_\beta$. Equations (6) and (15) are simultaneous inhomogeneous and homogeneous integral equations corresponding to same kernel $G_\alpha^\pm\, V^\alpha$. Equations (13) and (14) are in similar situations for the channel β. From the nature of the above derivations, using (7) to (15), it is to be noted that $|\psi_\alpha^+(E_\alpha)\rangle$, the solution of the inhomogeneous equation (6), defines the same function as that of the homogeneous equations (13) for all $\beta \neq \alpha$, and that happens as a consequence of Lippmann's identity (11). In the following discussion, we rewrite the homogeneous equations (13) and (15) as

$$|\psi^{\pm}_{\alpha' h\beta}(E_\alpha)\rangle = G^{\pm}_{\beta}(E_\alpha)\, V^{\beta}|\psi^{\pm}_{\alpha' h\beta}(E_\alpha)\rangle \tag{16}$$

$$|\psi^{\pm}_{\beta' h\alpha}(E_\alpha)\rangle = G^{\pm}_{\alpha}(E_\alpha)\, V^{\alpha}|\psi^{\pm}_{\beta' h\alpha}(E_\alpha)\rangle \tag{17}$$

to emphasize that, $|\psi^{\pm}_{\alpha' h\beta}(E_\alpha)\rangle$ is a solution of a homogeneous integral equation, eq (16), but we may note, in addition, that the solution of (16) is assumed to give the same function as the solution of (13) and also of (6):

$$|\psi^{\pm}_{\alpha' h\beta}(E_\alpha)\rangle = |\psi^{\pm}_{\alpha}(E_\alpha)\rangle. \tag{18}$$

It is further seen that, if $|\psi^{\pm}_{\alpha}(E_\alpha)\rangle$ is a particular solution of (6), so is $|\Phi^{\pm}_{\alpha}(E_\alpha)\rangle$ where

$$|\Phi^{\pm}_{\alpha}(E_\alpha)\rangle = |\psi^{\pm}_{\alpha}(E_\alpha)\rangle + \sum_{\beta \neq \alpha} C_\beta |\psi^{\pm}_{\beta, h\alpha}(E_\alpha)\rangle \tag{19}$$

the summation over β in (19) being for all channels other than α. Equation (19) is analogous to (3) and is the customary statement of the problem of non-uniqueness of the solutions Lippmann–Schwinger equation (6). It is important to remember that the results Fredholm theory of integral equation is not straightaway applicable to the integral equation of scattering (6) and (15), for their kernels $G^{\pm}_{\alpha}(E_\alpha)\, V^{\alpha}$ are not Fredholm type. Thus, although (19) is a consequence of (6) and (17), the result of compatibility condition corresponding to (4) of Fredholm theory, cannot be written down right away. We can, however, proceed to obtain a necessary condition for the coexistence of homogeneous and inhomogeneous integral equations. This is a pertinent question to ask for, in the present situation, $|\psi^{+}_{\alpha}(E_\alpha)\rangle$ in a three-particle system, obeys one inhomogeneous integral equation of the type (6) for kernel $G^{+}_{\alpha}(E_\alpha)\, V^{\alpha}$ and two different homogeneous integral equations of the type (13), with a different kernel $G^{+}_{\beta}(E_\alpha)\, V^{\beta}$ for $\beta \neq \alpha$, so the question of its being overdetermined vis-a-vis compatibility of equations concerned is to be looked into.

From (17), one would by complex conjugation write, for $\beta \neq \alpha$:

$$\langle \psi^{\pm}_{\beta' h\alpha}(E_\alpha)| = \langle \psi^{\pm}_{\beta' h\alpha}(E_\alpha)|\, V^{\alpha} G^{\mp}_{\alpha}(E_\alpha) \tag{20}$$

so that

$$\langle \psi^{-}_{\beta' h\alpha}(E_\alpha)|\, V^{\alpha}\psi^{+}_{\alpha}(E_\alpha)\rangle = \langle \psi^{-}_{\beta' h\alpha}(E_\alpha)|\, V^{\alpha} G^{+}_{\alpha}(E_\alpha)|\, V^{\alpha}\psi^{+}_{\alpha}(E_\alpha)\rangle \tag{21}$$

Further, using (6), we may write

$$\langle \psi^{-}_{\beta' h\alpha}(E_\alpha)|\, V^{\alpha}\psi^{+}_{\alpha}(E_\alpha)\rangle = \langle \psi^{-}_{\beta' h\alpha}(E_\alpha)|\, V^{\alpha}\phi_{\alpha}(E_\alpha)\rangle + \langle \psi^{-}_{\beta' h\alpha}(E_\alpha)|\, V^{\alpha} G^{+}_{\alpha}(E_\alpha)\, V^{\alpha}|\psi^{+}_{\alpha}(E_\alpha)\rangle \tag{22}$$

from (21) and (22) we have

$$\langle \psi^{-}_{\beta' h\alpha}(E_\alpha)|\, V^{\alpha}\phi_{\alpha}(E_\alpha)\rangle = 0 \tag{23}$$

and using (18), we rewrite (23) to obtain ($\beta \neq \alpha$) for all E_α

$$T^{-}_{\beta\alpha}(E_\alpha) = \langle \psi^{-}_{\beta}(E_\alpha)|\, V^{\alpha}|\phi_{\alpha}(E_\alpha)\rangle = 0. \tag{24}$$

A word of caution is perhaps necessary when operation of complex conjugation is performed on terms which are capable of contributing finitely from surface integrals at infinity, as shown by the following: Using Schrodinger equation for

$$|\psi_\alpha^+(E_\alpha)\rangle : (E_\alpha - \mathbf{H}_\alpha)|\psi_\alpha^+(E_\alpha)\rangle = V^\alpha|\psi_\alpha^+(E_\alpha)\rangle \tag{25}$$

we may write:

$$\langle\xi|\,V^\alpha\psi_\alpha^+(E_\alpha)\rangle = \langle\xi|(E_\alpha - \mathbf{H}_\alpha)|\psi_\alpha^+(E_\alpha)\rangle \tag{26}$$

$$= \langle(E_\alpha - \mathbf{H}_\alpha)\xi|\psi_\alpha^+(E_\alpha)\rangle - \mathscr{I}(\langle\xi|, |\psi_\alpha^+(E_\alpha)\rangle \tag{27}$$

where the surface integral $\mathscr{I}(\langle\psi_1|, |\psi_2)$ is defined as

$$\mathscr{I}(\langle\psi_1|, |\psi_2\rangle) = \langle\psi_1|\mathbf{H}_0\psi_2\rangle - \langle\mathbf{H}_0\psi_1|\psi_2\rangle. \tag{28}$$

Here $(\mathbf{H}_0\psi_1)$ means H_0 operating ϕ_1 etc. (Gerjuoy [5]) where H_0 is the kinetic energy operator and hence the integral on the RHS of (28) can be converted into surface integral at infinity by Green's theorem. Using (27) we can write

$$\langle\psi^-_{\beta' h\alpha}(E_\alpha)|\,V^\alpha\psi_\alpha^+(E_\alpha)\rangle = \langle\,V^\alpha\psi^-_{\beta' h\alpha}(E_\alpha)|\psi_\alpha^+(E_\alpha)\rangle$$
$$- \mathscr{I}(\langle\psi^-_{\beta, h\alpha}(E_\alpha)|, |\psi_\alpha^+(E_\alpha)\rangle. \tag{29}$$

Substituting for $\psi_\alpha^+(E_\alpha)$ on LHS in (29), using (6), and $\psi^-_{\beta' h\alpha}(E_\alpha)$ on RHS, using (15), we have

$$\langle\psi^-_{\beta' h\alpha}(E_\alpha)|\,V^\alpha\phi_\alpha(E_\alpha)\rangle + \langle\psi^-_{\beta' h\alpha}(E_\alpha)|\,V^\alpha G_\alpha^+(E_\alpha)\,V^\alpha\psi_\alpha^+(E_\alpha)\rangle$$
$$= \langle\,V^\alpha G_\alpha^-(E_\alpha)\,V^\alpha\psi^-_{\beta' h\alpha}|\psi_\alpha^+(E_\alpha)\rangle - \mathscr{I}(\langle\psi^-_{\beta' h\alpha}(E_\alpha)|, |\psi_\alpha^+(E_\alpha)\rangle. \tag{30}$$

Since $(G_\alpha^-)^+ = G_\alpha^+$ and V^α is hermitian, $(V^\alpha G_\alpha^-(E_\alpha)\,V^\alpha)^+ = (V^\alpha G_\alpha^+(E_\alpha)\,V^\alpha)$ Glöckle [1] we have, instead of (23):

$$\langle\psi^-_{\beta' h\alpha}(E_\alpha)|\,V^\alpha\phi_\alpha(E_\alpha)\rangle = -\mathscr{I}(\langle\psi^-_{\beta' h\alpha}(E_\alpha)|, |\psi_\alpha^+(E_\alpha)\rangle) \tag{31}$$

From (31), using (18), we get the following expression for the scattering matrix for all $\beta \neq \alpha$,

$$T^-_{\beta\alpha}(E_\alpha) = \langle\psi_\beta^-(E_\alpha)|\,V^\alpha\phi_\alpha(E_\alpha)\rangle = -\mathscr{I}(\langle\psi_\beta^-(E_\alpha)|, |\psi_\alpha^+(E_\alpha)\rangle) \tag{32}$$

The appearance of the extra surface integrals in (29) to (32) may be noted in comparison with the corresponding equations in (21) to (24). Now, for rearrangement scattering $\beta \neq \alpha$, each particle in the system is bound either to the target or to the projectile both initially as well as finally, so that the surface integral (32) at infinity is zero for all $\beta \neq \alpha$ Gerjuoy [5, 6]. Hence we have again,

$$T^-_{\beta\alpha}(E_\alpha) = \langle\psi_\beta^-(E_\alpha)|\,V^\alpha\phi_\alpha(E_\alpha)\rangle = 0 \tag{33}$$

for all $\beta \neq \alpha$. Equation (33), therefore, follows as a "necessary condition" whenever the inhomogeneous eqn. (6) co-exist with the homogeneous eqn. (15) for the case of rearrangement scattering $\beta \neq \alpha$.

Since

$$T^{(-)}_{\beta\alpha} = T^+_{\beta\alpha},\ T^+_{\beta\alpha}(E_\alpha) = \langle\phi_\beta|\,V^\beta|\psi_\alpha^+(E_\alpha)\rangle = 0 \tag{34}$$

on the energy shell, we conclude that the vanishing of the on-shell transition matrix $T^+_{\beta\alpha}(E_\alpha)$ for rearrangement scattering at arbitrary energy E_α is a "necessary condition" for the coexistence of the homogeneous and inhomogeneous equation of LS-type corresponding to same Kernel. Since the rearrangement scattering matrix $T^+_{\beta\alpha}(E_\alpha)$ cannot vanish identically for arbitrary energy E_α, we conclude that the homogeneous and inhomogeneous LS-equations, (6) and (15) (or (13) and (14)) cannot hold simultaneously. And since the inhomogeneous LS-equation (6) has some physical basis, the homogeneous LS-equation (15) either does not exist or admits only trivial solutions. This in turn cast doubts as to the validity of Lippmann's identity. It may be noted that the existence of a homogeneous integral equation does not necessarily mean that it has a non-trivial solution, in first place. Fredholm theory states that the homogeneous Fredholm integral equation (1) has only a finite number of solution Kolton and Kress [2]. But the corresponding exercise has not been done for (13) or (15). The way the homogeneous LS-equation are derived, by use of Lippmann's identity (11), the homogeneous integral equations (13) or (15) appear to have infinite number of solutions corresponding to every arbitrary E_α. For, similar algebraic operation lead to inhomogeneous and homogeneous equations, and same function $|\psi^\pm_\alpha(E_\alpha)\rangle$ (or $|\psi^\pm_\beta(E_\alpha)\rangle$) appear in (13) (or (14)) as well as in (6) (or (15)), assuming Lippmann's identity to be valid, and one would be tempted to assume that the non-trivial solutions of homogeneous LS-equation (15) give the same function given by solution of the inhomogeneous LS-equation (6).

In summary, we might say that the Lippmann's identity is infructuous as the homogeneous LS-equation it yields can only be permitted to have trivial solution and that the non-uniqueness problem of the solutions of LS-equations do not exist.

It may be noted that the above analysis carried out for three particle systems can be extended easily for many-particle systems without much changes in the text (see Ch. 3 of [1]).

In some of my earlier publications [3, 7–11] I had shown the defects in the treatment of the non-uniqueness problem by Gerjuoy [5] and others [1].

A brief discussion highlighting the defects in the arguments of [5] and [6] is presented below. The principal equations that we consider for the purpose are as follows:

$$(E_\alpha - \mathbf{H})|\psi^+_\alpha(E_\alpha)\rangle = 0 \tag{35}$$

$$(E_\alpha - \mathbf{H}_\alpha)|\phi_\alpha(E_\alpha)\rangle = 0 \tag{36}$$

$$|\psi^+_\alpha(E_\alpha)\rangle = |\phi_\alpha(E_\alpha)\rangle + |\chi^+_\alpha(E_\alpha)\rangle \tag{37}$$

$$(E_\alpha - \mathbf{H}_\alpha)|\psi^+_\alpha(E_\alpha)\rangle = (E_\alpha - \mathbf{H}_\alpha)|\chi^+_\alpha(E_\alpha)\rangle = \bar{V}_\alpha|\psi^+_\alpha(E_\alpha)\rangle \tag{38}$$

$$G^+_\alpha(E_\alpha)(E_\alpha - \mathbf{H}_\alpha) = (E_\alpha - \mathbf{H}_\alpha)G^+_\alpha(E_\alpha) = I \tag{39}$$

where I is the unity operator, $|\chi^+_\alpha(E_\alpha)\rangle$ is the scattered part of $|\psi^+_\alpha(E_\alpha)\rangle$. Multiplying (39) on the right by $|\chi^+_\alpha(E_\alpha)\rangle$ and (38) on the left by $G^+_\alpha(E_\alpha)$ and subtracting, Gerjuoy gets [5]:

$$\begin{aligned} G^+_\alpha(E_\alpha)\,\bar{V}_\alpha|\psi^+_\alpha(E_\alpha)\rangle - |\chi^+_\alpha(E_\alpha)\rangle &= G^+_\alpha(E_\alpha)\cdot(E_\alpha - \mathbf{H}_\alpha)|\chi^+_\alpha(E_\alpha)\rangle \\ - G^+_\alpha(E_\alpha)(E_\alpha - \mathbf{H}_\alpha)|\chi^+_\alpha(E_\alpha)\rangle &= -\mathscr{I}(G^+_\alpha(E_\alpha), |\chi^+_\alpha(E_\alpha)\rangle) \end{aligned} \tag{40}$$

Similar manipulations, with $|\psi_\alpha^+(E_\alpha)\rangle$, yields [6]:

$$G_\alpha^+(E_\alpha)\cdot \bar{V}_\alpha|\psi_\alpha^+(E_\alpha)\rangle - |\psi_\alpha^+(E_\alpha)\rangle = -\mathscr{I}(G_\alpha^+(E), |\psi_\alpha^+(E_\alpha)\rangle). \tag{41}$$

Similarly, with (36) and (39), Gerjuoy *et al* [6] deduce:

$$\mathscr{I}(G_\alpha^+(E_\alpha), |\phi_\alpha(E_\alpha)\rangle) = G_\alpha^+(E_\alpha)(E_\alpha - \mathbf{H}_\alpha)|\phi_\alpha(E_\alpha)\rangle - G_\alpha^+(E_\alpha)\cdot(E_\alpha - \mathbf{H}_\alpha)\cdot |\phi_\alpha(E_\alpha)\rangle \tag{42}$$

$$= |\phi_\alpha(E_\alpha)\rangle. \tag{43}$$

Now, using the definition of Green's function (9) and (10) we get

$$(Z - \mathbf{H}_\alpha)G_\alpha(Z) = G_\alpha(Z)(Z - \mathbf{H}_\alpha) = I \tag{43}$$

$$(Z - \mathbf{H})G(Z) = G(Z)(Z - \mathbf{H}) = I. \tag{44}$$

From the definition of surface integrals in (28) we have

$$F_\alpha(Z) = \mathscr{I}(G_\alpha(Z), |\phi_\alpha(E_\alpha)\rangle) \tag{45}$$

$$= G_\alpha(Z)(Z - \mathbf{H}_\alpha)|\phi_\alpha(E_\alpha)\rangle - G_\alpha(Z)(Z - \mathbf{H}_\alpha)|\phi_\alpha(E_\alpha)\rangle$$

$$= |\phi_\alpha(E_\alpha)\rangle - (Z - E_\alpha)\cdot G_\alpha(Z)|\phi_\alpha(E_\alpha)\rangle$$

$$= |\phi_\alpha(E_\alpha)\rangle - (Z - E_\alpha)\cdot \frac{1}{Z - E_\alpha}|\phi_\alpha(E_\alpha)\rangle \tag{46}$$

$$= 0 \text{ for } Z \neq E_\alpha. \tag{47}$$

clearly $F_\alpha(Z)$ as a well defined analytic function of the complex variable Z, vanishing identically everywhere in the complex Z-plane, excepting at points E_α on the real energy axis corresponding eigenvalues of H_α (the spectrum), where $F_\alpha(Z)$ is undefined, being of the indeterminate (see (46)) form 0/0. But $F_\alpha(Z)$ has a unique value zero as real axis is approached, from above or below:

$$\mathscr{I}(G_\alpha^+(E_\alpha), |\phi_\alpha(E_\alpha)\rangle) = \lim_{n\to 0} F_\alpha(E_\alpha + i\eta) = 0. \tag{48}$$

It is known that for suitable class of potentials, the coordinate representatives of Green's function $G_\alpha(Z)$ is an analytic function of Z in the entire Z-plane, excepting at the spectrum (the eigenvalues E_α) of H_α, so that the singularities of $G_\alpha(Z)$ are poles in the negative real axis, corresponding to boundstates of H_α, and a cut along the positive real axis, across which $G_\alpha(Z)$ is discontinuous, and that $G_\alpha(Z)$ becomes unbounded, when real axis is approached from above or below, being bounded as

$$\|G_\alpha(Z)\| \leqslant \frac{C}{|\mathscr{I}_m(Z)|} \tag{49}$$

(C is a constant). Hence $F_\alpha(Z)$ remains bounded as real axis is approached from above or below. Further, from (43), we have for $Z = E_\alpha + i\eta$:

$$(E_\alpha - \mathbf{H}_\alpha)G_\alpha(E_\alpha + i\eta) = G_\alpha(E_\alpha + i\eta)(E_\alpha - \mathbf{H}_\alpha) = I - i\eta\, G_\alpha(E_\alpha + i\eta) \tag{50}$$

so that, instead of (39), we have

$$(E_\alpha - \mathbf{H}_\alpha)G_\alpha^+(E_\alpha) = G_\alpha^+(E_\alpha)(E_\alpha - \mathbf{H}_\alpha) = I - \lim_{n\to 0} i\eta\, G_\alpha(E_\alpha + i\eta). \tag{51}$$

Since for all $\eta \neq 0$ we have $i\eta\, G_\alpha(E_\alpha + i\eta)|\phi_\alpha(E_\alpha)\rangle = |\phi_\alpha(E_\alpha)\rangle$ identically, we may take

$$P_\alpha(E_\alpha)\cdot|\phi_\alpha(E_\alpha)\rangle = \lim_{n\to 0} i\eta\, G_\alpha(E_\alpha + i\eta)|\phi_\alpha(E_\alpha)\rangle = |\phi_\alpha(E_\alpha)\rangle \tag{52}$$

Thus, curiously enough, we get from (51), (52) [8–11]:

$$(E_\alpha - \mathbf{H}_\alpha)G_\alpha^+(E_\alpha)|\phi_\alpha(E_\alpha)\rangle = G_\alpha^+(E_\alpha)(E - \mathbf{H}_\alpha)|\phi_\alpha(E_\alpha)\rangle = 0. \tag{53}$$

This result already shows that the general equation (43) for Green's function in the complex plane, defined as an inverse of the operator $(Z - H_\alpha)$, does not automatically lead to (39) as a defining for Green's function for real energies. Clearly, for arbitrary vector $|f\rangle$, which does not contain a constant multiple of $|\varphi_\alpha(E)\rangle$, we have

$$P_\alpha(E_\alpha)|f\rangle = 0 \tag{54}$$

so that (51) behaves the same way as (39) does. Then it is wrong to use (39) as universally valid. Stated otherwise, it is important to know the domain of an operator before using it, as mathematicians would insist. Gerjuoy fell into the trap by treating (39) as universally valid, even though he started from (43) and (44) (see (1.4) in [5]). As a result Gerjuoy *et al* ends up getting $|\phi_\alpha(E_\alpha)\rangle$ for the RHS of (43), instead of zero for it, obtained as in (48). Ironically, this erroneous result (43) emerged as a strong point in the criticism of Gerjuoy *et al* in either criticism against my work. They did not realize that in going from (42) to (44) they have actually used algebraic operations which are not permissible. For example, they have taken $E_\alpha G_\alpha^+(E_\alpha)|\phi_\alpha(E_\alpha)\rangle - E_\alpha G_\alpha^+ (E_\alpha)|\phi_\alpha, (E_\alpha)\rangle = 0$, which looks innocent but it actually involves division by zero, for $G_\alpha^+(E_\alpha)|\phi_\alpha(E_\alpha)\rangle = (1/0)|\phi_\alpha(E_\alpha)$ and hence the above cancellation is not a permissible step in mathematics ($\beta - \beta = 0$ provided β is an element of algebra which excludes β obtained through division by zero). This then makes the most important result (43) (eqn. (12) of [6]) meaningless and invalid, alongwith it their strong criticism (see last three para of [6]) of my work. They, instead, claimed to have trivially derived. $F_\alpha(E_\alpha) = |\phi_\alpha(E_0) > (= \mathscr{I}(E)$ in [6]) without realising that $F_\alpha(E_\alpha)$ cannot be "calculated" for E_α is a singular point of $F_\alpha(Z)$ and $G_\alpha(Z)$, and it is onlv meaningful to talk about $F_\alpha(E_\alpha + i\eta)$ for $\eta \to 0$ which remains well defined. Further $F_\alpha(E_\alpha + i\eta)$ is zero identically and not because of the exponentially decreasing boundary condition which they thought to be the case. In fact $G_\alpha(Z)$ of (43) is definable in abstract manner as the inverse of the operator $(Z - H_\alpha)$ without assumption about the behaviour of $G_\alpha(Z)$ at large distances.

Another questionable feature in Gerjuoy's analysis is the prescription [6] that

$$\mathscr{I}(G^+(E_\alpha), |\chi_\alpha^+(E_\alpha)\rangle) = 0 \tag{55}$$

$$\mathscr{I}(G_\alpha^+(E_\alpha), |\chi_\alpha(E_\alpha)\rangle) = 0 \tag{56}$$

In fact the mathematical content of these prescriptions were never presented in their paper and the arguments had only been handwaving in nature. It was originally [5] presented more as an appeal to the common sense – as something intuitively expected, that the scattered wave $|\chi_\alpha^+(E_\alpha)\rangle$ is everywhere outgoing and hence (55) was thought to the consequence which cannot but include (56). Thus they got rid of this terms from (40) to get the standard result for scattered wave. They have then essentially assumed (55) and (56) to be true initially [5], and later [6] taken (55) to be the definition of the term everywhere outgoing and though "reasonable to assume" that (56) to be true (see discussion following (15) in [6]).

It should however be noted in passing that (55) or (56) is not just one condition on scattered wave, but is actually an infinite number of conditions, as can be seen be converting the volume integral to surface integral by Green's theorem, integral should vanish in every direction on the surface. But then the important question that stands out is about the status/meaning of the solution of a second order differential equation (35) constrained to satisfy these infinite number of conditions. In fact the conditions such as (55) or (56) can only be satisfied if they are zero identically. Gerjuoy [5] or [6] is unperturbed about or perhaps do not appear to have realised the real implication of (55) or (56). It may be mentioned here that, for completely different reasons (55) and (56) are actually true, as proved [see eqs. (19) and (20) in [8]).

In view of the above, I consider the criticism of Gerjuoy *et al* [6] of my work as totally invalid and conclude that the non-uniqueness problem of the solutions of LS-equations for three or many particle systems does not exist.

References

[1] W Glockle, *The quantum mechanical few-body problem*, (Springer-Verlag, 1983)
[2] D Colton and R Kress, *in Integral equation methods in scattering theory*, (John Wiley, New York, 1983)
[3] S Mukherjee, *Pramana – J. Phys.* **34**, 173 (1990)
[4] R Courant and D Hilbert, *Methods of mathematical physics* (Academic, New York, 1963)
[5] E Gerjuoy, *Phys. Rev.* **109**, 1806 (1958)
[6] E Gerjuoy and S K Adhikari *Phys. Lett.* **A115** 1 (1986)
[7] S Mukherjee, *Phys. Lett.* **A81**, 207 (1981)
[8] S Mukherjee, *Phys. Lett.* **A83**, 1 (1981)
[9] S Mukherjee, *Pramana – J. Phys.* **16**, 81 (1981)
[10] S Mukherjee, *Pramana – J. Phys.* **18**, 495 (1982)
[11] S Mukherjee, *Pramana – J. Phys.* **27**, 623 (1986)
[12] M Reed and B Simon, *Methods of modern mathematical physics* (New York, Academic, 1979) Vol. I, III