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WKB approximation in complex time

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Abstract. The WKB approximation to the one particle Schrödinger equation in time is used to obtain the wavefunction at a given point as a sum of semiclassical terms, each corresponding to a different classical trajectory (real or complex) but ending up at the same point. A method to find out reflection coefficient for processes involving one and two turning points is developed and it is shown that the semiclassical complex analysis reproduces exactly the reflection coefficient that is obtained through the exact solution of the problem. The connection between pair production and reflection amplitude is also shown. The pair production amplitude in a time dependent gravitational background is calculated and it is shown that the vacuum considered in complex trajectory WKB analysis refers to adiabatic vacuum.

Keywords. WKB method; curved space time; particle production.

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1. Introduction

The use of WKB approximation in various physical problems dealing with turning points is well known [1]. Semiclassical scattering theory with complex trajectories has already been discussed by many [2–4]. The use of complex trajectory methods in one-dimensional problems was initiated by Stokes as early as 1857 [5] and more recently by many other workers [6–11]. All these methods deal with Schrödinger equation in space variable. However, our attempt is to generalize the result in time variable and to establish a connection between reflection coefficient and pair production amplitude. A preliminary report of the approach has already been made [12]. To justify the effectiveness of simple methods we calculate the pair production amplitude by the complex time reflection method and the standard method and show the equality of the results.

The Schrödinger type equations with the variable x replaced by t (in one-dimension) are generally encountered in many problems of physics. Particularly the temporal equation arising out of motion of scalar and spinor particle in a time dependent gravitational background [13–17] is of importance when one considers particle production in such a given space time. The problem of particle production in a time varying electromagnetic field has already been discussed [12]. The standard approach to see particle production in a gravitational background is through Bogolubov transformation technique [13, 15] or through Green function method [14]. In both the approaches the exact solution of the temporal equations are needed and the calculation of Green functions becomes very cumbersome. Not only that, the vacuum in curved space time of quantum field theory plays a delicate role. Here we encounter two types of vacuua namely adiabatic vacuum and conformal vacuum. Sometimes particle production occurs in conformal vacuum but not in adiabatic vacuum [13]. In our approach we find that the complex semiclassical method generates the adiabatic vacuum and in some special cases it may reduce to the conformal vacuum. We discuss a specific case to elaborate this aspect, others are only mentioned.

In § 2 we discuss the WKB approximation in complex time and calculate the reflection coefficient. In § 3 we discuss the connection between reflection in time and pair production to obtain

$$|\text{pair production amplitude}|^2 = |\text{reflection amplitude}|^2.$$

In § 4 we discuss the particle production scenario in two-dimensional Milne universe employing the complex time multiple reflection technique. The conclusions are summarized in § 5.

2. WKB approximation in complex time

Let us start with one dimensional Schrödinger equation in time,

$$\frac{d^2\psi}{dt^2} + (w^2 - V(t))\psi = 0, \tag{1}$$

where $V(t)$ is a potential term.

We now generalize the result of Schrempp and Schrempp [4] in complex t-plane where $V(t)$ is treated as a function of complex t. Equation (1) arises in many problems e.g. temporal equation in time dependent gravitational background dealing with motion of scalar and spinor particle, the Wheeler–DeWitt equation in quantum cosmology [18], transport equation in quark-gluon-plasma based on Wigner representation [19] and the production of particles in a time-dependent electromagnetic field. The basic objective of this work is to study such problems within the framework of complex trajectory semiclassical WKB approximation. We have successfully solved the above problems for the calculation of particle production amplitude e.g., particle production in gravitational background, e^+e^- pair production in time dependent strong electric field and quark-antiquark production amplitude in quark-gluon-plasma. In quantum cosmology we have calculated the wavefunction of the universe identifying the pair production wavefunction as wormhole solutions in Euclidean space. Here we restrict ourselves to pair production in a gravitational case where standard results are available for comparison.

The WKB approximation for complex trajectories accounts for contribution of the order of $\exp(-c/\hbar)$ to the usual WKB wave which would vanish in the classical limit. The WKB approximation generalized to complex time will turn out to reproduce quantitatively all quantum mechanical effects and remain accurate in cases when the potential $V(t)$ varies rapidly over a distance of wavelength or even less. In this approach one starts at a real point and ends at another real point, but is complex in between. In our work we consider two cases with

$$\Omega(t) = (w^2 - V(t))^{1/2}, \tag{2}$$

having one and two turning points, real or complex. The situation of two real turning points which is complex in between occurs in quantum cosmology in which $\Omega(t = a(t))$ becomes complex between two points $a = 0$ and $a = H^{-1}$ [18]. The standard WKB approximation with real t is written as

$$\psi = \frac{C_1}{[\Omega(t)]^{1/2}} \exp \int i\Omega(t) \mathrm{d}t + \frac{C_2}{(\Omega(t))^{1/2}} \exp \int (-i)\Omega(t) \mathrm{d}t \tag{3}$$

In complex t, we define

$$\int_{t_1}^{t} \Omega(t) \mathrm{d}t \equiv S(t, t_1) = \int_{t_1}^{t} (w^2 - V(t))^{1/2} \mathrm{d}t, \tag{4}$$

and the turning points are determined from

$$\Omega(t) = (w^2 - V(t))^{1/2} = 0. \tag{5}$$

The positive aspect of our method is that we do not have to find out the exact solution or the Green functions that are needed in standard calculations [13–15]. We have to evaluate simple integrals of type (4) involving branch points (i.e., turning points). The boundary conditions are chosen such that

$$\psi \sim \exp(iS(t, t_1)), \quad t \to -\infty, \tag{6}$$

$$\psi \sim \exp[iS(t, t_1)] + R \exp[-iS(t, t_1)], \quad t \to \infty, \tag{7}$$

where R is called the reflection amplitude. When we consider the pair production as a consequence of reflection in time, the reflection coefficient will be identified as pair production amplitude. This will be exemplified in later sections.

We now calculate the reflection coefficient for processes having one and two turning points.

Case A. One turning point

Let us consider a situation where (1) reduces to the form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + a^2(t - t_0)\psi = 0, \tag{8}$$

where the turning point is now at $t = t_0$ which is taken to be complex.

In complex semiclassical approximation the wavefunction ψ is written as (see figure 1(a), (b))

$$\psi(t) \underset{t \to \infty}{\sim} \exp[iS(t, t_1)] + (-i)\exp[iS(t_0, t_1)]\exp[-iS(t, t_0)], \tag{9}$$

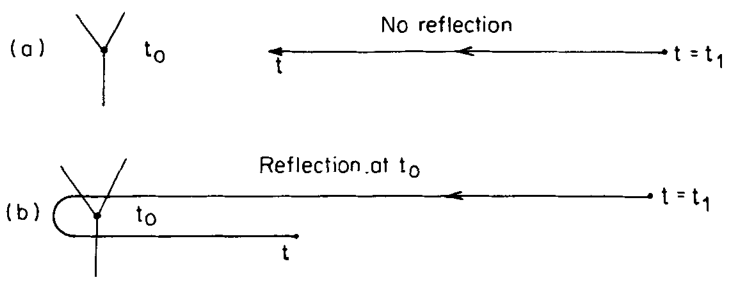

Figure 1.

where we have dropped the WKB pre-exponential factor for convenience. Equation (9) is interpreted as follows. A wave starting from $t = t_1$ and ending at t can be considered to be composed of (i) a wave starting at $t = t_1$, moving left to right, reaches at t and (ii) a wave starting at $t = t_1$, reaches the complex reflection point t_0 and turns back to t.

In (9), we have

$$S(x_1, x_2) = \int_{x_2}^{x_1} \Omega(t)\mathrm{d}t = \int_{x_2}^{x_1} [a^2(t - t_0)]^{1/2}\mathrm{d}t, \tag{10}$$

and the factor $(-i)$ in the second term of (9) has been introduced to ensure reflection at t_0. Using

$$\begin{aligned} S(t, t_0) &= S(t, t_0) + S(t_0, t_1) - S(t_0, t_1), \\ &= S(t, t_1) - S(t_0, t_1) \end{aligned} \tag{11}$$

we write (9) as

$$\psi(t) \sim \exp[iS(t, t_1)] - i\exp[2iS(t_0, t_1)\exp[-iS(t, t_1)]. \tag{12}$$

Comparing (7) and (12), we identify the reflection coefficient as

$$R = -i\exp[2iS(t_0, t_1)]. \tag{13}$$

To calculate pair production probability one needs $|R|^2$. Hence

$$|R|^2 = \exp[-4\,\mathrm{Im}\,S(t_0, t_1)], \tag{14}$$

where,

$$S(t_0, t_1) = \int_{t_1}^{t_0} \Omega(t)\mathrm{d}t. \tag{15}$$

Let us now proceed to calculate the reflection coefficient for a system having two turning points.

Case B. Two turning points

Let us consider the equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (\lambda + t^2)\right]\psi(t) = 0, \tag{16}$$

(16) has now two complex turning points at $t = \pm i\lambda^{1/2}$.

Here we have to consider repeated reflections between the turning points $t_2 = i\lambda^{1/2}$ and $t_1 = -i\lambda^{1/2}$. It may be noted that (16) is also obtained for the most general type of potentials. Near about the potential minima, (1) can be reduced to the form (16), expanding $V(t)$ into a Taylor series near about the minimum.

In the present case the resulting multiple reflection series is given by [2, 3, 4, 12] (See figure 2(a–c))

$$\begin{aligned} \psi(t) \underset{t \to \infty}{\rightarrow} &\exp[+iS(t, t_0) - i\exp[-i\{S(t_0, t_1) - S(t_1, t)\}] \\ &\times \sum_{\mu=0}^{\infty} [-i\exp\{iS(t_1, t_2)\}]^{2\mu} \end{aligned} \tag{17}$$

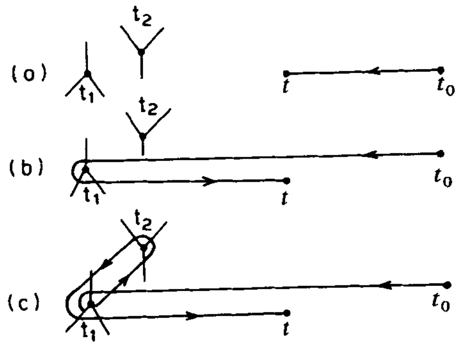

Figure 2. (a) No reflection (b) One reflection at t_1 (c) Reflections at t_1, t_2 and t_1.

$$= \exp[+iS(t,t_0)] - \frac{i\exp[2iS(t_1,t_0)]}{1+\exp[2iS(t_1,t_2)]}\exp[-iS(t,t_0)], \tag{18}$$

where

$$\sum_{\mu=0}^{\infty}[-i\exp\{iS(t_1,t_2)\}]^{2\mu} = \frac{1}{1+\exp[2iS(t_1,t_2)]}. \tag{19}$$

The interpretation of (17) is as follows. The classical trajectories building up the quantum mechanical wave are the direct (real) trajectory from t_0 to t represented by the first term in (17) and the trajectory from t_0 returning to t after 'complex' reflections between t_1 and t_2 leading to the geometrical series (19). In the usual WKB approximation with real semiclassical path we have only the first term $\exp[-iS(t,t_0)]$ in (17). The reflection coefficient for this one-dimensional problem is given by

$$\psi(t,t_0) \underset{t\to\infty}{\sim} \exp(-iwt) + R\exp(iwt), \tag{20}$$

such that,

$$R = -\frac{i\exp[2iS(t_1,t_0)]}{1+\exp[2iS(t_1,t_2)]}. \tag{21}$$

Let us now evaluate $S(t_1,t_0)$ and $S(t_1,t_2)$ identifying $t_2 = -i\lambda^{1/2}$ and $t_1 = i\lambda^{1/2}$. We find

$$S(t_1,t_0) = -i\pi\lambda/4 + \delta(t_0),$$
$$S(t_2,t_1) = +i\pi\lambda/2, \tag{22}$$

where $\delta(t_0)$ is a real function of t_0. Hence from (21),

$$|R|^2 = \frac{\exp(-\pi\lambda)}{[1+\exp(-\pi\lambda)]^2}. \tag{23}$$

This is the reflection amplitude according to complex time WKB approximation.

3. Connection between reflection in time and pair production

The connection between the reflection mechanism in time and the pair production phenomena is elaborated now. In fact the reflection coefficient R is identified as the

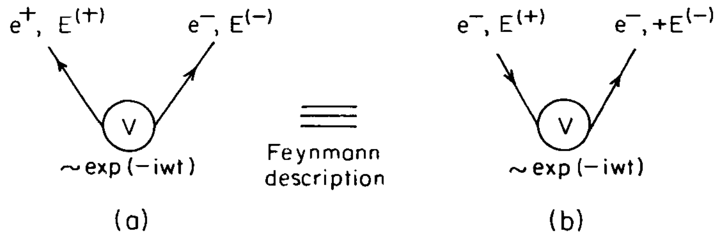

Figure 3.

pair production amplitude in the complex time multiple reflection technique. If the time variable t were to be replaced by the space variable x then R would have been referred to as the scattering amplitude.

Let us consider the pair production of e^+e^- by a potential $V(t) \sim \exp(-iwt)$. This is shown in figure 3.

According to Feynman prescription we know that negative energy particle solutions propagating backward in time $\equiv$ positive energy antiparticle solutions propagating forward in time. Thus figure 3(b) is the Feynman description of e^+e^- pair creation corresponding to figure 3(a). This motivates us to interpret figure 3(b) as a negative energy electron, moving backward in time, suffers a reflection at the potential site and after reflection becomes a positive energy electron moving forward in time. This idea to consider pair production as a process of reflection was considered by Biswas and Das [12]. The same idea was also pursued by Cornwall and Ticktopoulos [20] in a different context. They also consider the pair production as Klein paradox like situation, not in space but in time. According to Cornwall and Ticktopoulos [20], pair production corresponds to a reflection process in time in which there is no particle present at $t \to -\infty$ but at $t \to +\infty$ there is a particle moving forward in time and an antiparticle moving backward in time.

In this section we calculate the pair production amplitude for the two cases A and B of §2 using generalized WKB approximation [13, 21], that are frequently used in curved space time dependent problem.

We first take up case A of §2 and refer to equation (8). Let us introduce the dependent variable

$$\psi = \frac{F}{W^{1/2}}, \tag{24}$$

and the independent variable

$$q = \int_{t_1}^{t} w(t')\mathrm{d}t'. \tag{25}$$

Equation (8) then becomes

$$\frac{\mathrm{d}^2F}{\mathrm{d}q^2} + \left[1 + \frac{3}{4}\left(\frac{\dot{w}}{w^2}\right)^2 - \frac{1}{2}\left(\frac{\dot{w}}{w^3}\right)\right]F = 0 \tag{26}$$

where

$$\dot{w} = \frac{\mathrm{d}w}{\mathrm{d}t}.$$

Asymptotically, i.e., as $t \to \pm\infty$, the terms within the bracket in (26) tends to unity and F is of the form

$$F = A(q)\exp(iq) + B(q)\exp(-iq). \tag{27}$$

To ensure pair production, as mentioned earlier, we take the boundary conditions as

$$\begin{aligned} &\text{(i)} \quad A = 1, \quad B = 0 \quad \text{for } t \to -\infty, \\ &\text{(ii)} \quad A, B \neq 0 \quad \text{for } t \to +\infty. \end{aligned} \tag{28}$$

So at $t \to +\infty$, the amount of pair production is measured by $|B/A|^2$.

Evaluating the bracketted terms in (26) with respect to q and using $w(t) = a(t - t_0)^{1/2}$, we get from (26)

$$\frac{\mathrm{d}^2}{\mathrm{d}q^2}F + \left[1 + \frac{5}{36}\frac{1}{(q - q_0)^2}\right]F = 0, \tag{29}$$

where

$$q_0 = \int_{t_1}^{t_0} w(t)\mathrm{d}t. \tag{30}$$

Under the substitution

$$F = (q - q_0)^{1/2} \quad U(q - q_0),$$

we have

$$U'' + \frac{1}{q - q_0}U' + \left[1 - \frac{(1/3)^2}{(q - q_0)^2}\right]U = 0 \tag{31}$$

The solution of (31) is the Hankel function $H^{(1,2)}_{1/3}(q - q_0)$, so that we write

$$F(q) = (q - q_0)^{1/2} H^{(1,2)}_{1/3}(q - q_0) \tag{32}$$

Using

$$\begin{aligned} H^{(1)}_n(x) &\xrightarrow{x \to \infty} \left(\frac{2}{\pi x}\right)^{1/2} \exp i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\ H^{(2)}_n(x) &\xrightarrow{x \to \infty} \left(\frac{2}{\pi x}\right)^{1/2} \exp i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \end{aligned} \tag{33}$$

and equation (27), (28) and (30) one finds,

$$|R|^2 = |B/A|^2 = \exp(-4\,\mathrm{Im}\,q_0). \tag{34}$$

Equation (34) is exactly the same as (14) which was obtained from complex path analysis having one turning point.

We now refer to (16) for the case B in §2 to find out $|R|^2$ obtaining the exact solutions which can be written in terms of parabolic cylinder functions D_ν. The results are [22]

$$\psi(t) \propto D_{-1/2 + i\lambda/2[-(1-i)\tau]}, \tag{35}$$

$$= (2t^2)^{-1/4}\exp\left\{\frac{\pi}{8}(\lambda + i) + \frac{i\lambda}{4}\ln 2 + \frac{it^2}{2} + \frac{i\lambda}{2}\ln(-t)\right\} \quad \text{for } t \to -\infty, \tag{36}$$

$$= \frac{(2\pi)^{1/2}}{\left[\frac{(1-i\lambda)}{2}\right]}(2t^2)^{-1/4}\exp\left\{-\frac{\pi}{8}(\lambda - i) - i\frac{\lambda}{4}\ln 2 - \frac{it^2}{2} - i\frac{\lambda}{2}\ln t\right\}$$

$$+ (2t^2)^{-1/4}\exp\left\{-\frac{3}{8}\pi(\lambda + i) + \frac{i\lambda}{4}\ln 2 + i\frac{t^2}{2} + i\frac{\lambda}{2}\ln t\right\}$$

$$\text{for } t \to +\infty. \quad (37)$$

Writing (37) in the form (27) and identifying $|R| = |B/A|$, we get

$$|R| = \frac{\exp\left(-\frac{\pi\lambda}{2}\right)\Gamma\left[\frac{(1-i\lambda)}{2}\right]\Gamma\left[\frac{(1+i\lambda)}{2}\right]}{2\pi}$$

Thus

$$|R|^2 = \frac{\exp(-\pi\lambda)}{[1+\exp(-\pi\lambda)]^2}. \quad (38)$$

Again we find the same result as (23).

Obviously we find that the complex time reflection technique provides an effective method of calculation. The application of this technique in various problems will be dealt with in detail elsewhere. We however give an illustration in §4.

4. An application

In this section we briefly outline the result obtained from the method of complex multiple reflection in time when applied to the particle production scenario in 2-dimensional Milne universe. We have also applied this technique with remarkable success in the particle production mechanism in a strong time dependent electromagnetic field [12]. Other instances will be dealt with elsewhere.

In Kasner's space time the metric is given by

$$ds^2 = dt^2 - t^{2a_1}dx^2 - t^{2a_2}dy^2 - t^{2a_3}dz^2,$$

where a_1, a_2 and a_3 are real numbers and $a_1 + a_2 + a_3 = a_1^2 + a_2^2 + a_3^2 = 1$. In such a space time we find out the temporal equation obtained from the Dirac equation

$$-i\tilde{\gamma}^\mu \psi_{;\mu} + m\psi = 0 \quad (39)$$

with $a_1 = a_3 = 0$ and $a_2 = 1$ to be

$$\left[\partial_0^2 + \left(a + \frac{mT_0}{t^2}\right)\right]\bar{f}_I = 0, \quad (40)$$

where

$$T_0 = \frac{k_2}{k_1^2 + k_3^2}\begin{pmatrix} k_1 & -k_3 \\ -k_3 & -k_1 \end{pmatrix}, \quad (41)$$

$$a = m^2 + k_1^2 + k_3^2 \quad (42)$$

and

$$\psi = t^{-1/2}(2\pi)^{-3/2}\exp[i\mathbf{k}\cdot\mathbf{x}]\begin{pmatrix} f_I \\ f_{II} \end{pmatrix}, \quad (43)$$

with

$$f_I = \exp\left[\frac{1}{2}\int \frac{t'-t_0}{t'} \frac{\partial}{\partial t'} \frac{t'}{(t'-t_0)} \mathrm{d}t'\right] \bar{f}_I. \tag{44}$$

The turning points are at $t_{1,2} = \mp iM$ where $M = \left(\frac{mT_0}{a}\right)^{1/2}$. We now calculate $S(t_2, t_1)$ and $S(\infty, t_1)$ obtaining

$$|R|^2 = \frac{\exp[-2\pi(mT_0)^{1/2}]}{[1+\exp\{-2\pi(mT_0)^{1/2}\}]^2}. \tag{45}$$

This shows that there is particle production in Kasner's space time. One interesting aspect of this result is that we have considered here the space time as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - t^2 \mathrm{d}y^2 - \mathrm{d}z^2, \tag{46}$$

which reduce to Milne universe when $x = z = 0$. In Milne universe the standard results are available for comparison [13]. To get $|R|^2$ for Milne universe from (44) we note that we have to put $k_1 = k_3 = 0$. In that case $mT_0 \to \infty$, $(m \neq 0)$ thereby implying that $|R|^2 \to 0$.

This result is in conformity with the standard result [13] that there is no particle production in two dimensional Milne universe corresponding to adiabatic vacuum. However for $m \to 0$ we have $mT_0 \to k_2$ and in this case this adiabatic vacuum turns out into a conformal vacuum where there will be particle production. This fact is also corroborated by Birrel and Davies [13].

5. Conclusion

In this paper we have developed a method of calculating the particle production amplitude using complex semiclassical WKB method and have shown that the resulting wavefunction is identical to that obtained by generalized WKB approximation. We have verified the results obtained by our method for several cases with the corresponding standard results. The equality of the results also establish the Feynman's prescription on a solid footing and the reflection in time as a useful method for seeking the pair production. In Minkowski space time the calculation of R refers to Minkowski vacuum which is well defined. But in curved space time the vacuum is not so well defined. Here we have two types of vacuum known as conformal vacuum and adiabatic vacuum. In some cases there is particle production in conformal vacuum but not in adiabatic vacuum. In curved space time the standard method to calculate the pair production amplitude is through Bogolubov transformation technique [13]. It is thus necessary to clarify the role of privileged vacuum by some other method. The present work exemplifies this situation. It should be pointed out that WKB wavefunction obtained through complex semiclassical approach or through generalized WKB approximation coincides with the exact solution at large time. This in turn implies that we are dealing with adiabatic vacuum. That our construction selects the adiabatic vacuum follows from the fact that we have started from the WKB approximation which itself is valid in the adiabatic limit. To be specific we note that Green's function $G_1^{\mathrm{WKB}}(x, x')$ obtained through generalized WKB approximation i.e., through complex trajectory semiclassical method reproduces Hadamard singularities in the propogator [23]. We then consider WKB wavefunction to be a good candidate for the vacuum which itself is an adiabatic vacuum by definition. This aspect will be dealt elsewhere in future. While working on pair production in curved space time one should rely on adiabatic vacuum compared to the conformal vacuum except when conformal vacuum is dictated by the symmetry of the problem (e.g., two dimensional Milne universe).

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