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Astrophysically significant solutions of the Einstein and Einstein–Maxwell equations from the Laplace's seed

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Abstract. The stationary solutions given by Amenedo and Manko generated from known solutions of Laplace's equation as seed have been generalised to include the electromagnetic field. Further, the exterior solution of an axially symmetric rotating body with higher multipole moments and a solution corresponding to a Kerr object embedded in a gravitational field are given. We also give a method for constructing stationary vacuum solutions from static magnetovac solutions and vice versa and discuss a specific application of this method.

Keywords. General theory of relativity; Einstein/Einstein Maxwell solutions.

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1. Introduction

The relativistic effect is important in describing the behaviour of real astrophysical objects. Even in the dimension of the solar system the effect cannot be ignored. Consequently, several workers since Einstein's times were seriously engaged in constructing well behaved solutions of Einstein or Einstein–Maxwell equations which describe the actual field inside or outside an astrophysical object.

The line element was chosen for a desired symmetry and then Einstein/Einstein–Maxwell field equations were written down in advantageous coordinates for obtaining solutions. Naturally only simple solutions were discovered in such cases due to the high non-linearity of the field equations. This conventional method became impertinent with the discovery of the Kerr/Kerr–Newman metric in early sixties. The discovery of the Kerr metric also raised fresh hopes among astrophysicists since the metric so discovered turned out to be a probable solution representing physically the field of a rotating astrophysical object. But it is now believed that the Kerr metric cannot represent the external field of an arbitrary realistic star because of a very special relationship between its multiple moments and angular momentum [1].

Later, the reformulation of the Einstein/Einstein–Maxwell field equations was tried and in some cases interesting solutions including the Kerr/Kerr–Newman were rediscovered [2–5]. In the meantime, the hidden symmetry of the relativistic field equations came to light and different transformation techniques [6, 7] of generating new solutions from the old were discovered one after another, almost putting an end to the conventional methods of solving the non-linear relativistic field equations directly. Thus the analysis, interpretation or selection of well behaved realistic solutions became increasingly difficult.

Among the many methods which came into existence, was the inverse-scattering method, discovered by Belinskii and Zakharov [8] and the other a transformation technique of HKX (1979) ([9]). These two methods not only rediscovered the already known solutions of the Einstein equations but promised more realistic solutions representing the exact field of the astrophysical objects.

On the other hand, with the development of astrophysics we find a proliferation of suspected black holes. Black holes have been suspected to be present in several X-ray binaries both in our galaxy and in the neighbouring ones. Quasars have long been suspected to contain supermassive black holes in their cores. In addition, more and more galactic nuclei including our own are being suspected to harbour black holes (For a review see Shapiro and Teukolsky [10]). Nowhere do these black holes exist in isolation but are surrounded by hot accretion discs of matter. The high temperature produces ionization and the galactic magnetic field may also be present. So the rotating black holes may be distorted under the action of both electromagnetic and gravitational fields of surrounding matter. Wagh and Dadhich [11] tried to study the effects of current loops and weak electromagnetic fields on rotating black holes by the method of superposition. They investigated the size of the ergosphere and the possibility of extraction of energy by Penrose process. But since the gravitational field near a black hole is strong the correctness of the method of superposition is doubtful, if we remember that Einstein's equations are highly non-linear. Therefore exact solutions representing distorted black holes in external gravitational or electromagnetic fields were discovered [12–20] and their geometrical properties were studied [21–24].

About seventy five years ago, Weyl [25] showed how all static axially symmetric solutions of Einstein's vacuum field equations might be constructed from solutions of Laplace's equations. Recently Amenedo and Manko [1] and Quevedo and Mashhoon [26] with the aid of HKX transformation have given simple methods of obtaining the axially symmetric stationary vacuum solutions of the Einstein's equations from the solution of Laplace's equation and a series of papers have already been published by them illustrating the viability of their elegant methods with a lot of new solutions as perquisitive. (See references in Manko and Khakimov [27]).

In § 1, we summarize the method given by Manko *et al.* Assuming that Kerr metric does not correspond to the exact field of rotating arbitrary stars, in § 2 we give two sets of new axially symmetric stationary vacuum solutions generated from solutions of Laplace's equation as seed. Both of them reduce to the Kerr metric for suitable values of the parameters and thus are more general than Kerr metric. The first one is asymptotically flat and represents the field of a rotating axially symmetric source with higher multipole moments. The second solution is not asymptotically flat and may be interpreted as the gravitational field due to a rotating object embedded in an external gravitational field. For a particular choice of the constants it reduces to the metric of Kerns and Wild [16] which is interpreted as a Schwarzschild metric embedded in a gravitational field. Both the solutions are not only for a better description of the field of a deformed mass but for a better description of the gravitational field of rotating star which is spherically symmetric in the static limit.

In § 3 we give a method for generating stationary vacuum solutions from static magnetovac solutions and vice versa simply by inspection. With its help a magnetostatic metric is constructed which has Melvin's massless geon as a special case. In § 4 we give a method for constructing the stationary solution of Einstein–Maxwell equations corresponding to the stationary gravitational metric of Amenedo and Manko [1] and § 5 gives conclusions.

1. Method of Manko *et al.*

The axially symmetric stationary line element is written as

$$ds^2 = K^2 f^{-1}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\varphi^2\right]$$
$$-f(dt-\omega d\varphi)^2, \tag{1}$$

where x and y are the prolate spheroidal coordinates related to the Papapetrou coordinates

$$\rho^2 = \sigma^2(x^2-1)(1-y^2),$$
$$Z = \sigma xy. \tag{2}$$

The Einstein field equations for metric (1) were reformulated by Ernst [2, 3] and showed that the problem of solving metric (1) rests mainly on a single equation in terms of $\mathscr{E}$, the Ernst complex potential, and the metric functions f, γ and w can then be constructed easily.

$$\mathscr{E} = f + i\phi, \ (\mathscr{E}+\mathscr{E}^*)\nabla^2\mathscr{E} = 2\nabla\mathscr{E}\cdot\nabla\mathscr{E}$$
$$\Phi_x = k^{-1}(x^2-1)^{-1}f^2 w_y, \ \Phi_y = -k^{-1}(1-y^2)^{-1}f^2 w_x, \tag{3}$$

$$\gamma_x = \frac{1}{4(x^2-y^2)}\left[\frac{1-y^2}{f^2}\{x((x^2-1)f_x^2-(1-y^2)f_y^2)-2y(x^2-1)f_x f_y\}\right.$$
$$\left.-\frac{f^2}{x^2-1}\{x((x^2-1)w_x^2-(1-y^2)w_y^2)-2y(x^2-1)w_x w_y\}\right] \tag{4}$$

$$\gamma_y = \frac{1}{4(x^2-y^2)}\left[\frac{x^2-1}{f^2}\{y((x^2-1)f_x^2-(1-y^2)f_y^2)+2x(1-y^2)f_x f_y\}\right.$$
$$\left.-\frac{f^2}{1-y^2}\{y((x^2-1)w_x^2-(1-y^2)w_y^2)+2x(1-y^2)w_x w_y\}\right],$$

where Φ is known as the twist potential, ∇^2, the Laplacian and ∇ the gradient operators respectively.

Gutsunaev and Manko [28] obtained a solution for Ernst potential $\mathscr{E}$, as a combination of a solution 2ψ of the Laplace's equation (5), with the solutions for a and b, derived from two pairs of first order differential equations (7).

$$(x^2-1)\psi_{xx}+(1-y^2)\psi_{yy}+2x\psi_x-2y\psi_y = 0, \tag{5}$$

$$\mathscr{E} = e^{2\psi}\frac{x(1+ab)+iy(b-a)-(1-ia)(1-ib)}{x(1+ab)+iy(b-a)+(1-ia)(1-ib)}, \tag{6}$$

$$(x-y)a_x = 2a[(xy-1)\psi_x+(1-y^2)\psi_y],$$
$$(x+y)b_x = -2b[(xy+1)\psi_x+(1-y^2)\psi_y],$$
$$(x-y)a_y = 2a[-(x^2-1)\psi_x+(xy-1)\psi_y],$$
$$(x+y)b_y = -2b[-(x^2-1)\psi_x+(xy+1)\psi_y]. \tag{7}$$

Thus, at least in principle, any solution, 2ψ of Laplace's equation (5) will render a solution for functions a and b from the two pairs of first order differential equations (7). $\mathscr{E}$ is then determined from (6).

Amenedo and Manko [1] succeeded in integrating all the equations responsible for the construction of f, γ and w, from general solution of (2ψ) and a and b. The results are:

$$f = e^{2\psi}\frac{(x^2-1)(1+ab)^2-(1-y^2)(b-a)^2}{\{(x+1)+(x-1)ab\}^2+\{a(1+y)+b(1-y)\}^2}, \tag{8}$$

$$= e^{2\psi}AB^{-1}(\text{say}),$$

$$w = 2k\,e^{-2\psi}CA^{-1}+k_2, \tag{9}$$

$$e^{2\gamma} = k_1(x^2-1)^{-1}e^{2\gamma'}A, \tag{10}$$

where γ' is the γ function of the static metric with Laplace's solution as

$$\psi' = \tfrac{1}{2}\ln[(x-1)(x+1)^{-1}]$$

and

$$c = (x^2-1)(1+ab)[(b-a)-y(a+b)]+(1-y^2)(b-a)[1+ab+x(1-ab)], \tag{11}$$

where, k, k_1 and k_2 are constants.

The stationary metric (1) representing a stationary gravitational field is thus completely solved just from any desired solution of the Laplace's equation. In fact, this is an achievement of Amenedo and Manko. Although the HKX method is of the same status, practically speaking, a more complicated algebraic computation is associated there. It is not as simple as Amenedo and Manko when applied for.

2. Application of the method

Amenedo and Manko, using different solutions of the Laplace's equation constructed the axially symmetric stationary vacuum metrics different from each other and published a series of papers on the same. It is well known that different Laplace's solutions can be extracted from the general solution of a type,

$$2\psi = \sum_{n=1}^{\alpha}(-1)^n q_n Q_n(x)\cdot P_n(y), \tag{12}$$

(where P_n and Q_n are respectively the Legendre polynomials and the associated Legendre functions of the second kind) by cutting down desired portion from the summation. One need not always extend the summation up to infinity. Amenedo and Manko's solution (8)–(11) applies both for a portion of (12) or for the entire. This is one of the elegant aspects of Amenedo and Manko's work [1].

We have taken a simple particular solution of the Laplace's equation (5), not directly from equation (12) and constructed a new axially symmetric stationary metric which contains some of the well-known metrics, as special cases.

1st set: We have taken,

$$2\psi = \alpha_0(x+y)^{-1}, \tag{13}$$

where α_0 is a constant and solved for the functions a and b with the help of the differential equations given in (7).

$$a = -\alpha \exp[\alpha_0 y(x+y)^{-1}], \tag{14}$$

$$b = \alpha \exp[-\alpha_0(1+xy)(x+y)^{-2}] \tag{15}$$

(α is an arbitrary constant).

It is now a simple algebraic compilation which constructs f and w from (8), (9) and (11). The function γ however, remains to be determined since determination of γ is a separate task.

$$e^{2\gamma} = k_1 A(x^2-y^2)^{-1} \exp\left[\frac{\alpha_0}{4}(x+y)^{-4}(1-y^2)\{4(x+y)^2 - \alpha_0(x^2-1)\}\right] \tag{16}$$

where k_1 is a new constant.

The metric functions f, w and γ are, no doubt very complicated due to the presence of exponential terms in the functions a and b. After a cumbersome algebraic computation we have revealed their behaviour at spatial infinity i.e. at $x \to \infty$.

$$e^{2\psi} = 1 + \frac{\alpha_0}{x} + \frac{\alpha_0(\alpha_0 - 2y)}{2}\frac{1}{x^2} + O(\gamma^{-3})\ldots \tag{17}$$

$$f = 1 + \frac{\alpha_0(1-\alpha^2) - 2(1+\alpha^2)}{1-\alpha^2}\cdot\frac{1}{x} + \frac{\alpha_0^2(1-\alpha^2)^2 - 4(1+\alpha^2)^2 - 4\alpha_0(1-\alpha^4)}{2(1-\alpha^2)\cdot x^2} + O(\gamma^{-3}) \tag{18}$$

$$w = \frac{4k\alpha(1+\alpha^2)(1-y^2)}{(1-\alpha^2)^2}\frac{}{x} - \frac{4K\alpha\alpha_0}{1-\alpha^2}, \tag{19}$$

$$a = -\alpha\left[1 + \frac{\alpha_0 y}{x} + \frac{\alpha_0(\alpha-2)y^2}{2}\frac{}{x^2}\ldots\right] \tag{20}$$

$$b = \alpha\left[1 - \frac{\alpha_0 y}{x} + \frac{\alpha_0}{2x^2}\{(\alpha_0+4)y^2 - 2\}\right] \tag{21}$$

$$e^{2\gamma} = 1 + \frac{2\alpha_0\alpha^2(1-\alpha^2) - (1+\alpha^2)}{(1-\alpha^2)^2}\frac{1}{x^2} + \ldots. \tag{22}$$

The above asymptotic expansions of $e^{2\psi}$, f, w, a, b, and $e^{2\gamma}$ show that they are asymptotically flat at large distances i.e. at $x \to \infty$.

A further coordinate transformation,

$$Kx = r - m, \quad y = \cos\theta, \tag{23}$$

shows that the mass of the source described by (8)–(10) is:

$$M = \frac{K\{2(1+\alpha^2) - \alpha_0(1-\alpha^2)\}}{2(1-\alpha^2)}. \tag{24}$$

The metric contains the Kerr metric, which can be shown by putting $\alpha_0 = 0$ and redefining the constant α as

$$p = (1-\alpha^2)(1+\alpha^2)^{-1}, \quad q = 2\alpha(1+\alpha^2)^{-1}, \tag{25}$$

so that $p^2 + q^2 = 1$. The asymptotic expansion of f and w containing new constants p and q then becomes,

$$f = 1 - \frac{2k}{p}\cdot\frac{1}{r} + O(r^{-2})\ldots \tag{26}$$

$$w = \frac{2k^2 q}{p^2}\cdot\frac{1}{r} + O(r^{-2}) \tag{27}$$

This gives the mass M and angular momentum J of the metric for $\alpha_0 = 0$ as kp^{-1} and $(k^2 qp^{-2})$ respectively such that

$$J = \mathrm{M}^2 q. \tag{28}$$

This is a characteristic ratio of the Kerr metric. Its conversion into the Schwarzschild metric is now a trivial task. One may put $q = 0$ to eliminate rotation parameter and get the Schwarzschild.

The metric given by (8)–(10) with (13)–(15) is very complicated and its complete analysis deserves a separate work with the help of a computer. However, a preliminary investigation shows that the metric is singular at the poles $x = \pm 1$, $y = \pm 1$ and the space time region exterior to hypersurface $x = 1$ is also not free from singularities. This is partly justified by the fact that, on the equatorial plane $y = 0$, there exist more than one singular points at $x > 1$. The exact position of singularities depends upon the value of α. The seed solution is singular on the surface $x + y = 0$ and this singularity remains intact in the derived metric functions (8)–(10). But this singular surface remains hidden within the hypersurface $x = 1$. When $\alpha_0 = 0$, the metric reduces to Kerr metric for specified value of α. The singular points or surfaces of the derived metric do not exactly coincide with Kerr metric and are thus called distorted singularity [29]. The distortion is due to the superposition of a Kerr metric with a seed (13). Amenedo *et al* [1] succeeded to ascertain the nature of singularities of their metric with the help of computer algebra system Reduce 3.3. The nature of singularities of our metric (regular, naked or mildly naked) is a problem for future work and no definite conclusion can be arrived at this moment.

Thus our metric so derived in (8)–(10) with (13)–(15) is new and reduces to the Kerr/Schwarzschild metric when proper adjustment of the parameters is made. The metric then may be interpreted as a Kerr metric superposed non-linearly with a gravitational field represented by the term $e^{\alpha_0 (x+y)^{-1}}$.

2nd set: Here we have taken another simple solution of the Laplace's equation

$$2\psi = \alpha_0 xy, \tag{29}$$

and completed the construction of the metric function f, w and γ as usual with a and b given by

$$a = -\alpha \exp[\alpha_0 z_0 (x-y)],$$
$$b = \alpha \exp[-\alpha_0 z_0 (x+y)]. \tag{30}$$

$$f = \frac{e^{\alpha_0 xy}[(x^2-1)(e^{\alpha_0 z_0 y} - \alpha^2 e^{-\alpha_0 z_0 y})^2 - \alpha^2(1-y^2)(e^{\alpha_0 z_0 x} + e^{-\alpha_0 z_0 x})^2]}{\{(x+1)e^{\alpha_0 z_0 y} - \alpha^2(x-1)e^{-\alpha_0 z_0 y}\}^2 + \alpha^2\{(1-y)e^{-\alpha_0 z_0 x} - \alpha^2(1+y)e^{\alpha_0 z_0 x}\}^2} \tag{31}$$

$$w = 2k e^{-\alpha_0 xy} C A^{-1} + k_2, \tag{32}$$

$$A = e^{-2\alpha_0 z_0 y}[(x^2-1)(e^{\alpha_0 z_0 y} - \alpha^2 e^{-\alpha_0 z_0 y})^2 - \alpha^2(1-y^2)(e^{\alpha_0 z_0 x} + e^{-\alpha_0 z_0 x})^2] \tag{33}$$

$$C = \alpha e^{z_0 \alpha_0 y}[(x^2-1)(e^{\alpha_0 z_0 y} - \alpha^2 e^{-\alpha_0 z_0 y}) \times \{e^{-\alpha_0 z_0 x} + e^{\alpha_0 z_0 x} - y(e^{-\alpha_0 z_0 x} - e^{\alpha_0 z_0 x})\} + (1-y^2)(e^{-\alpha_0 z_0 x} + e^{\alpha_0 z_0 x})\{e^{\alpha_0 z_0 y} - \alpha^2 e^{-\alpha_0 z_0 y} + x(e^{\alpha_0 z_0 y} + \alpha^2 e^{-\alpha_0 z_0 y})\}] \tag{34}$$

$$e^{2\gamma} = k_1 A(x^2-y^2)^{-1} e^{\alpha_0 y} - \frac{\alpha_0^2}{4}(x^2-1)(1-y^2), \tag{35}$$

where α_0, α, z_0, k, k_1 and k_2 are arbitrary constants, which may be adjusted to convert the constructed metric into other known metrics.

For example:

i) One may put $\alpha_0 = \alpha = 0$ and get the Schwarzschild metric in prolate spheroidal coordinates,

$$f_s = (x-1)(x+1)^{-1}, \quad w_s = 0, \quad e^{2\gamma} = k_1(x^2-1)(x^2-y^2)^{-1}. \tag{36}$$

ii) When $\alpha_0 \neq 0$ but $\alpha = 0$, we obtain the Kerns and Wild metric [16]

$$\begin{aligned} f_{kw} &= (x-1)(x+1)^{-1}\exp(\alpha_0 xy), \\ w_{kw} &= 0, \\ e^{2\gamma} &= k_1(x^2-1)(x^2-y^2)^{-1}\exp[\alpha_0 y - \frac{\alpha_0^2}{4}(x^2-1)(1-y^2)]. \end{aligned} \tag{37}$$

This metric (37) is interpreted as a Schwarschild metric embedded in a gravitational field. The nature of the metric and its singularities are discussed in Kerns and Wild [16].

iii) Kerr metric can be got if one puts $\alpha_0 = 0$ but $\alpha \neq 0$ and redefines α as $p = (1-\alpha^2)(1+\alpha^2)^{-1}$ and $q = 2\alpha(1+\alpha^2)^{-1}$

$$\begin{aligned} f_k &= (p^2x^2 + q^2y^2 - 1)[(px+1)^2 + q^2(1-y^2)]^{-1}, \\ w_k &= 2k[q(1-y^2)(px+1)][p(p^2x^2+q^2y^2-1]^{-1}. \end{aligned} \tag{38}$$

iv) with no restriction on the constants i.e. $\alpha_0 \neq 0$ and $\alpha \neq 0$ the solution in its general form given by (31)–(35), represents the Kerr metric embedded in a gravitational field if one takes the line of interpretation as that of Kerns and Wild. According to Manko's interpretation, it may also be said that the metric contained in (31)–(35), is a non-linear superposition of the Kerr metric with a gravitational field represented by the term $e^{\alpha_0 xy}$. Amenedo and Manko [1] discovered an analogous solution with the above interpretation but their metric does not reduce to the known Kerns–Wild metric in the static limit whereas ours does.

The parameter α_0 is connected with the strength of the external gravitational field and α with that of Kerr parameters p and $q \cdot \alpha_0$ affects the Kerr body to become more elongated only along its axis entailing an increment in the polar circumference keeping the equatorial circumference constant. The metric is singular at the poles $x = \pm 1$, $y = \pm 1$ and more than one singular points exist on the equatorial plane $y = 0$.

3. Stationary gravitational to static magnetovac

In this section a transformation similar to Bonnor (1966) is restated. By Bonnor's transformation one gets static electrovac solution (i.e. containing both electric and magnetic components of the scalar electromagnetic potential) from a stationary vacuum solution and one more transformation is needed to cut down the electrostatic part from the scalar electromagnetic potential if one desires to retain only the magnetovac solution. It escaped our attention so far that a stationary metric directly goes over to a static magnetovac metric simply by a Bonnor-like transformation. This saves a lot of labour for further computation. In the following we briefly show the formal similarity among the field equations of stationary vacuum and static magnetovac metric in the Papapetrou coordinates.

An axially symmetric stationary metric is represented by

$$ds^2 = e^{u}(dt - w d\varphi)^2 - e^{-iu}[e^{2\gamma^s}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \tag{39}$$

and the corresponding vacuum field equations are

$$u_{\rho\rho} + u_{zz} + \frac{u_\rho}{\rho} = -\rho^{-2} e^{2u}(w_\rho^2 + w_z^2),$$

$$w_{\rho\rho} + w_{zz} - \frac{w_\rho}{\rho} = -2(u_\rho w_\rho + u_z w_z), \tag{40}$$

$$2\gamma^s_\rho = \frac{\rho}{2}(u_\rho^2 - u_z^2) - \frac{\rho^{-2}}{2} e^{2u}(w_\rho^2 - w_z^2),$$

$$2\gamma^s_z = \rho[u_\rho u_z - \rho^{-2} e^{2u} w_\rho w_z], \tag{41}$$

where the notations have their usual meaning.

On the other hand an axially symmetric static metric is given by:

$$ds^2 = e^{2\beta} dt^2 - e^{2(\gamma^m - \beta)}(d\rho^2 + dz^2) - e^{-2\beta}\rho^2 d\varphi^2, \tag{42}$$

and the magnetovac field equations read:

$$\beta_{\rho\rho} + \beta_{zz} + \frac{\beta\rho}{\rho} = \rho^{-2} e^{2\beta}(A_{3,z}^2 + A_{3,\rho}^2),$$

$$A_{3,\rho\rho} + A_{3,zz} - \frac{A_{3,\rho}}{\rho} = -2(\beta_\rho A_{3,\rho} + \beta_z A_{3,z}), \tag{43}$$

$$\gamma^m_\rho = \rho[(\beta_\rho^2 - \beta_z^2) + \rho^{-2} e^{2\beta}(A_{3,\rho}^2 - A_{3,z}^2),$$

$$\gamma^m_z = 2\rho[\beta_\rho \beta_z + \rho^{-2} e^{2\beta} A_{3,\rho} A_{3,z}], \tag{44}$$

where A_3 is the true magnetic component of the electromagnetic 4-potential and related to the pseudo magnetic potential A_3' as w is related to twist potential Φ in Ernst's notation. A close look at the two different sets of field equations (40)–(41) and (43)–(44) reveals that the magnetovac solutions of the static case may be constructed from the stationary one and vice versa via the transformation

$$\beta = u, \quad A_3 = iw, \quad \gamma^m = 4\gamma^s, \tag{45}$$

where superscripts m and s refer to the magnetovac and stationary cases respectively. The imaginary quantity 'i' is eliminated by the parameter change technique of Das [30, 31].

We now apply the above theorem to construct the static magnetovac solution from the stationary solution derived in the first set of § 2.

$$\begin{aligned} e^{2\beta} &= \left[e^{2\psi} \frac{(x^2-1)(1-ab)^2 + (1-y^2)(b-a)^2}{\{(x+1)-(x-1)ab\}^2 - \{a(1+y)+b(1-y)\}^2} \right]^2, \\ A_3 &= 2k\,e^{-2\psi}[(x^2-1)(1-ab)\{(b-a)-y(a+b)\} \\ &\quad + (1-y^2)(b-a)\{(1-ab)+x(1+ab)\}] \\ &\quad \times [(x^2-1)(1-ab)^2 + (1-y^2)(b-a)^2]^{-1} + k_1, \\ e^{2\gamma^m} &= k_1^4 A^4 (x^2-y^2)^{-4} \exp[\alpha_0(x+y)^{-4}(1-y^2)\{4(x+y)^2 \\ &\quad - \alpha_0(x^2-1)\}], \end{aligned} \tag{46}$$

where functions a and b remain the same as given in (14)–(15).

From the asymptotic behaviour of $e^{2\beta}$ and A_3 we obtained the mass of the source (M) and magnetic dipole moment (μ).

$$M = \frac{k\{2(1-\alpha^2) - \alpha_0(1+\alpha^2)\}}{(1+\alpha^2)} \tag{47}$$

$$\mu = \frac{4k^2\alpha(1-\alpha^2)}{(1+\alpha^2)^2}. \tag{48}$$

Under the condition,

$$\alpha^2 = (2-\alpha_0)(2+\alpha_0)^{-1}, \tag{49}$$

the solution goes over to the one describing the field of a massless magnetic dipole. The solution given in (46) does not however reduce to the Schwarzschild field when the magnetic field is switched off by putting $\alpha = 0$. This result is consistent with Bonnor's electrovac metric [32] which he obtained from the Kerr solution.

The theorem described in this section applies to the T-S classes of stationary solutions too. For other classes, sometimes more operations are needed [33], but the parameter change technique does not lose its merit there too.

4. Electrovac generalization of the Manko's stationary metric

It is shown in § 2 as to how we obtain an axially symmetric stationary vacuum metric, using Amenedo and Manko's prescription from an arbitrary solution of the Laplace's equation, simply by solving two pairs of the first order differential equations given in (7). The next generalization of their stationary gravitational metric is, no doubt, its electromagnetic analogue, which in principle, can be constructed by one of the methods discovered [2, 3, 34, 35]. Each of the methods has its own merit over the other. Here we have chosen Ernst's procedure, in the form applied by one of us (Das and Banerji [36]).

In the following we give the procedure of constructing the electromagnetic generalization of the vacuum stationary gravitational metric (1) of Amenedo and Manko with solutions given in (8)–(11): the Ernst potential $\mathscr{E}_0$ of which, has been written in (6). The other Ernst potential ξ_0 is defined as

$$\xi_0 = \frac{1+\mathscr{E}_0}{1-\mathscr{E}_0} = \frac{u+iv}{m+in}(\text{say}) \tag{50}$$

and it was shown by Ernst [2, 3] that the ξ function of the electromagnetically generalized metric is not much different from ξ_0

$$\xi = (1-qq^*)^{1/2}\xi_0 = \mu\xi_0, \tag{51}$$

where $\mu = (1-qq^*)^{1/2}$, q is a complex quantity associates with the electromagnetic scalar potential Φ.

$$\Phi = q\,e^{i\rho}(\mu\xi_0 + 1)^{-1}, \tag{52}$$

where ρ is a phase factor and

$$qq^* < 1. \tag{53}$$

The f and γ function of the electromagnetic generalized metric can be constructed mechanically with the help of the technique of Das and Banerji [36], but the construction of w requires much of labour and intuition. In fact, the whole problem concentrates on this point. As we do not know the exact form of 2ψ, a or b, but wish to integrate some functions connecting them, the difficulties are encountered. However we are able to solve it. The technique followed, owes its genesis to the work of Das and Banerji [36], Yamazaki [37] and Quevedo and Mashhoon [26].

$$f_e = \frac{u^2+v^2-m^2-n^2}{\left(u+\dfrac{m}{\mu}\right)^2 + \left(v+\dfrac{n}{\mu}\right)^2} \tag{54}$$

$$\gamma_e = \gamma, \tag{55}$$

where e, a suffix, is used to denote the metric of the electromagnetically generalized case. The complete metric functions for the electromagnetic generalization of the metric of Amenedo and Manko read

$$\begin{aligned} f_e = {} & [(x^2-1)(1+ab)^2 - (1-y^2)(b-a)^2] \\ & \times \left[\left(\frac{1+\mu}{2\mu}\right)^2 \{(1+ab)|(x+1)^2 + ab(x-1)^2| \right. \\ & + (b-a)|b(1-y^2) - a(1+y^2)|\}\, e^{-2\psi} \end{aligned}$$

$$+\left(\frac{\mu-1}{2\mu}\right)^2\{(1+ab)|(x-1)^2+ab(x+1)^2|$$

$$+(b-a)|b(1+y)^2-a(1-y)^2|\}\,e^{2\psi}$$

$$+\frac{\mu^2-1}{2\mu^2}\{(x^2-1)(1+ab)^2-(1-y^2)(b-a)^2\}]^{-1}, \tag{56}$$

$$\gamma_e=\gamma, \tag{55'}$$

$$\mu w_e=4\sigma\alpha\left(1+\frac{k}{2}\right)(1-\alpha^2)^{-1}-\sigma[(x^2-1)(1+ab)^2$$

$$-(1-y^2)(b-a)^2]^{-1}[e^{-2\psi}\left(2+\frac{k}{2}\right)\{(x^2-1)(1+ab)|$$

$$|b-a-y(b+a)|+(1-y^2)(b-a)|1+ab+x(1-ab)|\}$$

$$+\frac{k}{2}e^{2\psi}\{(x^2-1)(1+ab)|b-a+y(b+a)|$$

$$+(1-y^2)(b-a)|1+ab-x(1-ab)|\}], \tag{57}$$

$$\mu=(1-qq^*)^{1/2},\quad k=\frac{(\mu-1)^2}{\mu},$$

$$k_1=4\sigma\alpha(1-\alpha^2)^{-1},\quad k_2=(1-\alpha^2)^2. \tag{58}$$

$$\Phi=q\,e^{i\rho}(\mu\xi_0+1)^{-1} \tag{52}$$

$$\xi_0=[x(1+ab)(1+e^{2\psi})+(1-ab)(1-e^{2\psi})$$

$$+i\{y(b-a)(1+e^{2\psi})-(a+b)(1-e^{2\psi})\}]$$

$$\times[x(1+ab)(1-e^{2\psi})+(1-ab)(1+e^{2\psi})$$

$$+i\{y(b-a)(1-e^{2\psi})-(a+b)(1+e^{2\psi})\}]. \tag{59}$$

With the above, the construction of the new general stationary solution of Einstein–Maxwell equations is complete. We have chosen Ernst's procedure [2,3] of construction for the following reason. It is known that, in general, if the astrophysical objects be at all charged owing to a significant charge-separation developed during its evolution, the charge is expected to be very small. In Ernst's prescription [2,3]

$$q^2<1, \tag{53}$$

where $|q|$ is identified as the charge per unit mass of the source. Thus the solution of axially symmetric stationary Einstein–Maxwell equations constructed in (56)–(59) is consistent with the requirement of a real astrophysical object and the exact nature of the singularities in the metric depends on the nature of the seed function ψ taken.

The general electrovac metric generated here reduces to Amenedo and Manko's axially symmetric stationary gravitational metric when the electromagnetic field is withdrawn by making $\mu=1$ (i.e. $q=0$). For $\psi=0$, $q\neq0$, $a=-\alpha$ (constant) and $b=\alpha$, the general metric goes over to Kerr–Newman metric in prolate spheroidal coordinates x and y. A further coordinate transformation,

$$x=r-m,\quad y=\cos\theta \tag{60}$$

and rearrangement of the constants,

$$P = (1-\alpha^2)(1+\alpha^2)^{-1}, \quad Q = 2\alpha(1+\alpha^2)^{-1}$$

$$|q^2|^{1/2} = \mathrm{e}/m, \quad (m^2 - \mathrm{e}^2 - 1)/m^2 = a^2, \tag{61}$$

and

$$P^{-1} = m\mu,$$

casts the f-function of the metric into a conventionally recognised form of the Kerr-Newman metric

$$f_{kN} = (r^2 - 2mr + a^2\cos^2\theta + \mathrm{e}^2)(r^2 + a^2\cos^2\theta)^{-1}. \tag{62}$$

Now, its reduction to the Reissner–Nordstrom metric is a trivial task and can be achieved by putting $a = 0$. In addition, when the charge parameter $e = 0$, we get the well-known Schwarzschild metric from it.

5. Conclusion

We have chosen a solution of the Laplace's equation different from the others and a new asymptotically flat axially symmetric stationary gravitational metric is constructed in the 1st set of § 2. This metric contains the mass term as well as some of the other mass multipole moments but not all possible multipole moments simultaneously. If one makes a choice of ψ as in (12), the presence of all possible mass moments is ascertained. The stationary vacuum metric constructed in the 2nd set is not asymptotically flat but for a particular adjustment of the constants it reduces to the metric of Kerns and Wild, which is interpreted as a Schwarzschild metric embedded in a gravitational field. The metric of the 2nd set may, thus, be interpreted as a stationary generalization of the Kerns and Wild metric.

In § 3, a new magnetostatic metric is constructed from the result of § 2 with a simple theorem stated there for the first time. This magnetostatic metric is also asymptotically flat and may be converted to a massless magnetic geon of Melvin. When magnetic field is switched off, one obtains the Weyl $\partial = 2$ static metric instead of the Schwarzschild ($\partial = 1$). This property is similar to the solution of Bonnor [7].

In § 4 an algorithm is presented which transforms the stationary gravitational metric of Amenedo and Manko into its electromagnetic analogue. The main problem of integrating equations for w_e is also solved there and one may obtain w_e from w just by inspection. The beauty of the generalized electrovac metric constructed in the last section is revealed by the fact that all the stationary gravitational metric obtained from the prescription of Amenedo and Manko [1], are merely the special cases of the electrovac generalization. On one side, the Kerr-Newman and Reissner–Nordstrom metric are included and on the other side the Kerr or the Schwarzschild is also contained in it. Moreover the said electromagetic generalization is not Bonnor like [7]; since one does not get the Schwarzschild metric in the static limit (also shown in § 3) from Bonnor's transformation applied to the Kerr, where as here, one may get easily the Schwarzschild metric as shown here.

A more detailed examination of physical properties of the newly reported metrics still remains a task for the future. The preliminary analysis gives us hope that the exact solutions possibly represent the exterior gravitational field of a rotating star and as such can exhibit interest for astrophysicists.

Astrophysically significant solutions

Although we have not been able, as yet, to construct the T–S, $\partial = 2, 3, 4$, etc. stationary solutions from the Laplace's solution with the help of the transformation of Manko and Coworkers or by the HKX transformation, we are in a position now to anticipate that it will become a reality in the near future. The transformation of Amenedo and Manko is worth mentioning and has already advanced a few steps towards the goal.

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