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Floquet states of a periodically kicked particle

A K SIKRI and M L NARCHAL
Physics Department, Punjabi University, Patiala 147002, India

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Abstract. We present explicit expressions for the Floquet states of a periodically kicked particle in coordinate and momentum representations. These states have been used to evaluate the energy of the particle after an arbitrary number of kicks.

Keywords. Floquet states; kicked rotator system.

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1. Introduction

The study of quantum systems with Hamiltonians periodic in time has attracted much attention in recent years [1–7]. This is because of the fact that similar studies with classical systems have led to the understanding of the phenomenon which is now referred to in literature as classical chaos. The most extensively studied quantum system is the periodically kicked rotator. Fishman *et al* [3] established the mapping between the periodically kicked rotator and the Anderson localization in disordered solids. The work on quantum kicked rotator has been summarized in a recent book by Haake [8] and is relevant whenever the configuration space of the system under consideration can be scaled or mapped to the finite domain $(0, 2\pi)$. This cannot be done in those cases where the configuration space of the system extends from $-\infty$ to ∞ or from 0 to ∞ [7]. A free particle is the simplest example of such a system.

Much work has appeared in literature in which the time evolution of periodically kicked systems has been studied using Floquet theory. This includes besides the extensively studied kicked rotator, systems such as Fermi accelerator [9, 10], Morse oscillator [11, 12], kicked spin system [13, 14] etc. The dynamics of the system kicked periodically is essentially governed by the one step Floquet operator which is an ordered product of the free Hamiltonian time development operator and the kick operator. It is now realized [6, 10] that the spectral properties of this operator largely determine the behaviour of the system. If the eigenstates of the Floquet operator, referred to as quasienergies, are discrete and the system is bounded, then it is quantum mechanically stable and reassembles itself infinitely often with the passage of time [15]. If the quasienergies are continuous then the system is expected to show diffusive growth in energy.

Apart from a few general results there is practically no work available in literature on the explicit evaluation of the Floquet states which ought to form the most acceptable and natural basis for studying the time evolution of the periodically kicked systems. We present the precise form of quasienergies and quasistates for a kicked free particle. The outline of the paper is as follows: In § 2 we obtain the general expressions for the Floquet states in momentum and coordinate representations. These states are shown to be orthonormal and complete. This formalism is applied in § 3 to calculate the energy of a free particle after n kicks. Finally the results are summarized in § 4.

2. General expressions for Floquet states

Consider a free particle of mass m subject to time periodic Hamiltonian

$$H = H_0 - \alpha x \sum_n \delta(t - nT) \tag{1}$$

where

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{2}$$

and T is the kicking period. The dynamics of the system is described by one period evolution operator

$$U = \exp(i\alpha x T/\hbar)\exp(-iH_0 T/\hbar). \tag{3}$$

Here the first part describes the evolution due to kicks whereas the second part gives the evolution in between the kicks.

Substituting $k = \alpha T/\hbar$ we may write the Floquet operator U as

$$U = \exp(ikx)\exp[(i\hbar T/2m)(\partial^2/\partial x^2)]. \tag{4}$$

The Floquet states are the eigenstates of the operator U and are defined by the equation

$$U|\beta\rangle = \exp(-i\beta)|\beta\rangle \tag{5}$$

where $|\beta\rangle$ is the Floquet state having quasienergy β. We shall obtain the quasienergy states using the p representation. The Floquet state $|\beta\rangle$ may be written as

$$|\beta\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_\beta(\omega)\exp(i\omega x)\mathrm{d}\omega \tag{6}$$

where $\psi_\beta(\omega)$ is the space Fourier transform of $|\beta\rangle$.

Substituting the values of U and $|\beta\rangle$ in (5) we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_\beta(\omega - k)\exp[(-i\hbar T/2m)(\omega - k)^2]\exp(i\omega x)\mathrm{d}\omega$$
$$= \frac{\exp(-i\beta)}{2\pi}\int_{-\infty}^{\infty} \psi_\beta(\omega)\exp(i\omega x)\mathrm{d}\omega. \tag{7}$$

Thus $\psi_\beta(\omega)$ must satisfy the difference equation

$$\psi_\beta(\omega - k)\exp[(-i\hbar T/2m)(\omega - k)^2] = \exp(-i\beta)\psi_\beta(\omega). \tag{8}$$

Equation (8) can be easily shown to have the solution

$$\psi_\beta(\omega) = \exp[(-i\hbar T/2m)(a\omega^3 + b\omega^2 + c\omega)] \tag{9}$$

where

$$a = 1/3k, \tag{10}$$

$$b = -1/2, \tag{11}$$

$$c = -2m\beta/\hbar Tk + k/6. \tag{12}$$

Equation (12) shows that the parameter β may be arbitrarily chosen. Thus the quasienergies of a periodically kicked free particle form a continuum.

Substituting the value of $\psi_\beta(\omega)$ in (6), the expression for Floquet state $|\beta\rangle$ becomes

$$|\beta\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-if(\omega))\exp(i\beta\omega/k)\exp(i\omega x)\mathrm{d}\omega \tag{13}$$

where

$$f(\omega) = \frac{\hbar T}{2m}\left(\frac{\omega^3}{3k} - \frac{\omega^2}{2} + \frac{k\omega}{6}\right). \tag{14}$$

The normalized Floquet state is given by

$$|\beta\rangle = \frac{1}{2\pi\sqrt{k}}\int_{-\infty}^{\infty} \exp(-if(\omega))\exp(i\beta\omega/k)\exp(i\omega x)\mathrm{d}\omega. \tag{15}$$

The coordinate representation of Floquet state $|\beta\rangle$ turns out to be

$$|\beta\rangle = \frac{\exp(i\beta/2)}{3\sqrt{k}}\exp(ikx/2)z\left[J_{1/3}\left(\frac{2\gamma z}{3}\right) + J_{-1/3}\left(\frac{2\gamma z}{3}\right)\right] \tag{16}$$

where

$$z = (2\gamma mk/\hbar T)^{1/2} \tag{17a}$$

$$\gamma = \beta/k + \hbar kT/24m + x. \tag{17b}$$

It is easy to verify that the scalar product of Floquet states $|\beta\rangle$ and $|\beta'\rangle$ is

$$\langle\beta'|\beta\rangle = \delta(\beta' - \beta). \tag{18}$$

Thus we conclude that the Floquet states of a free particle belonging to different eigenvalues are orthonormal.

Further the states satisfy the completeness relation

$$\int_{-\infty}^{\infty} \mathrm{d}\beta|\beta\rangle_{x\,x'}\langle\beta| = \delta(x - x') \tag{19}$$

where $|\beta\rangle_x$ is the state defined by (15) and $|\beta\rangle_{x'}$ is the same state in which the coordinate variable x is replaced by x'.

Thus any arbitrary free particle state $|\psi\rangle$ may be expanded in terms of the complete set of Floquet states as

$$|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\beta|\beta\rangle\langle\beta|\psi\rangle \tag{20}$$

3. Expectation value of observables

Suppose ψ_0 and ψ_n are the wavefunctions of the system immediately after time 0 and $nT(n=1,2,3,4,\ldots)$ respectively. Then

$$\psi_n = U^n \psi_0. \tag{21}$$

Since the set $|\beta\rangle$ is complete, we expand the ket $|\psi_0\rangle$ as

$$|\psi_0\rangle = \int_{-\infty}^{\infty} \mathrm{d}\beta |\beta\rangle\langle\beta|\psi_0\rangle. \tag{22}$$

Substituting this value in (21) and using (5) we obtain

$$|\psi_n\rangle = \int_{-\infty}^{\infty} \mathrm{d}\beta \exp(-in\beta)|\beta\rangle\langle\beta|\psi_0\rangle. \tag{23}$$

If O is any operator then its expectation value in a state $|\psi_n\rangle$ after n kicks is

$$\langle O\rangle_n = \langle\psi_n|O|\psi_n\rangle.$$

Using (23) it takes the form

$$\langle O\rangle_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}\beta' \exp(-in(\beta-\beta'))\langle\beta|\psi_0\rangle\langle\psi_0|\beta'\rangle\langle\beta'|O|\beta\rangle \tag{24}$$

Energy of the kicked particle: The energy of the free particle after n kicks is given by

$$E_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\beta \mathrm{d}\beta' \exp(-in(\beta-\beta'))\langle\beta|\psi_0\rangle\langle\psi_0|\beta'\rangle\langle\beta'|H_0|\beta\rangle. \tag{25}$$

It can be easily shown that

$$\langle\beta'|H_0|\beta\rangle = \frac{\hbar^2}{4\pi mk}\int_{-\infty}^{\infty} \omega^2 \exp(i(\beta-\beta')\omega/k)\mathrm{d}\omega \tag{26a}$$

$$\langle\beta|\psi_0\rangle = \frac{1}{2\pi\sqrt{k}}\int_{-\infty}^{\infty} \exp(if(\omega))\exp(-i\beta\omega/k)\psi(\omega)\mathrm{d}\omega \tag{26b}$$

$$\langle\psi_0|\beta'\rangle = \frac{1}{2\pi\sqrt{k}}\int_{-\infty}^{\infty} \exp(-if(\omega'))\exp(i\beta'\omega'/k)\psi^*(\omega')\mathrm{d}\omega' \tag{26c}$$

where $\psi(\omega)$ is the Fourier transform of ψ_0.

Substituting these values in (25) and rearranging various terms we obtain

$$E_n = \frac{\hbar^2}{16\pi^3 mk^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\omega\,\mathrm{d}\omega'\,\mathrm{d}\omega''\,\psi^*(\omega')\psi(\omega)\omega''^2$$
$$\times \exp(i[f(\omega)-f(\omega')])\int_{-\infty}^{\infty} \mathrm{d}\beta \exp[-i\beta(n+(\omega/k)-(\omega''/k))]$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}\beta' \exp[i\beta'(n+(\omega'/k)-(\omega''/k))]. \tag{27}$$

Integrating over β and β' we obtain

$$E_n = \frac{\hbar^2}{4\pi m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\omega\,\mathrm{d}\omega'\,\mathrm{d}\omega''\,\psi^*(\omega')\psi(\omega)\omega''^2 \exp(i[f(\omega)-f(\omega')])$$
$$\times\,\delta(\omega-\omega''+nk)\delta(\omega'-\omega''+nk). \tag{28}$$

Integration over ω and ω' gives

$$E_n = \frac{\hbar^2}{4\pi m}\int_{-\infty}^{\infty} \omega''^2\psi^*(\omega''-nk)\psi(\omega''-nk)\mathrm{d}\omega''. \tag{29}$$

Changing the variable of integration from $\omega''-nk$ to ω, the above expression for E_n reduces to

$$E_n = \frac{\hbar^2}{4\pi m}\left[\int_{-\infty}^{\infty} \omega^2|\psi(\omega)|^2\mathrm{d}\omega + n^2k^2\int_{-\infty}^{\infty}|\psi(\omega)|^2\mathrm{d}\omega + 2nk\int_{-\infty}^{\infty}\omega|\psi(\omega)|^2\mathrm{d}\omega\right]. \tag{30}$$

It can be shown that

$$\int_{-\infty}^{\infty}|\psi(\omega)|^2\mathrm{d}\omega = 2\pi, \tag{31a}$$

$$\frac{\hbar^2}{4\pi m}\int_{-\infty}^{\infty}\omega^2|\psi(\omega)|^2\mathrm{d}\omega = \left\langle\frac{p_x^2}{2m}\right\rangle_0, \tag{31b}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega|\psi(\omega)|^2\mathrm{d}\omega = \frac{1}{\hbar}\langle p_x\rangle_0, \tag{31c}$$

where $\langle\;\rangle_0$ means the expectation value of observable in initial state ψ_0.

Using these results in (30) we obtain

$$E_n = \left\langle\frac{p_x^2}{2m}\right\rangle_0 + \frac{n^2\hbar^2k^2}{2m} + \frac{n\hbar k}{m}\langle p_x\rangle_0. \tag{32}$$

In terms of α it becomes

$$E_n = \left\langle\frac{p_x^2}{2m}\right\rangle_0 + \frac{\alpha^2T^2}{2m}n^2 + \frac{\alpha Tn}{m}\langle p_x\rangle_0. \tag{33}$$

This expression shows that the energy of particle grows monotonically with the number of kicks without showing any recurrence.

4. Summary

The explicit expressions for Floquet states of a periodically kicked free particle have been obtained in coordinate and momentum representations. The quasienergies of the system have been found to form a continuum. The momentum representation of Floquet states has been utilized to calculate the energy of particle after an arbitrary number of kicks. The energy of the particle has been found to increase monotonically with each kick, which is also expected from classical considerations. If the particle starts from rest classically then the energy increases quadratically with the number of kicks. The formalism is generally applicable to situations whose dynamics is governed by the repeated application of a one step time development operator.

References

[1] M V Berry, N L Balazs, M Tabor and A Voros, *Ann. Phys. (NY)* **122**, 26 (1979)
[2] R Blumel, R Meir and U Smilansky, *Phys. Lett.* **A103**, 353 (1984)
[3] S Fishman, D R Grempel and R E Prange, *Phys. Rev. Lett.* **49**, 509 (1982)
[4] D R Grempel, R E Prange and S Fishman, *Phys. Rev.* **A29**, 1639 (1984)
[5] F M Izrailev, *Phys. Rev. Lett.* **56**, 541 (1986)
[6] B Milek and P Seba, *Phys. Rev.* **A42**, 3213 (1990)
[7] R F Fox and B L Lan, *Phys. Rev.* **A41**, 2952 (1990)
[8] Fritz Haake, *Quantum signatures of chaos* (Springer–Verlag, 1991)
[9] J V Jose and R Cordery, *Phys. Rev. Lett.* **56**, 290 (1986)
[10] Petr Seba, *Phys. Rev.* **A41**, 2306 (1990)
[11] R Blumel and U Smilansky, *Phys. Rev. Lett.* **58**, 2531 (1987)
[12] John J Tanner and M Matti Maricq, *Phys. Rev.* **A40**, 4054 (1989)
[13] P W Milonni, J R Ackerhalt and M E Goggin, *Phys. Rev.* **A35**, 1714 (1987)
[14] T Geisel, *Phys. Rev.* **A41**, 2989 (1990)
[15] T Hogg and B A Huberman, *Phys. Rev. Lett.* **48**, 711 (1982)