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String cosmology in higher dimensional spherically symmetric space-time

SUBENOY CHAKRABORTY and ASHOK KR. CHAKRABORTY
Department of Mathematics, Jadavpur University, Calcutta 700032, India

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Abstract. Some cosmological solutions for string model are derived in higher dimensional spherically symmetric space-time, following the techniques used by Letelier. The equations of state for strings have been used for different solutions. Also polynomial relation between the metric coefficients has been assumed in some cases.

Keywords. String cosmology; higher dimension; spherically symmetric.

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1. Introduction

So far string cosmology has been discussed [1–5] for different metrics in the usual 4-dimensional space-time from classical point of view. String concept is important at the early stages of evolution before particle creation and solutions reveal that the particle dominated era followed string dominated era.

In recent years, study of cosmological phenomena [6–9] in higher dimension has been getting much interest due to requirement of space-time of more than 4-dimensions in super string theory [10] and Yang-Mills super gravity in its field theory limit. Although there is no direct observational evidence in favour of the existence of higher dimension, it is assumed that the universe could have preceded by a higher dimensional stage, which at later times becomes effectively 4-dimensional in the sense that the extradimensions become unobservably small due to dynamical contraction [11]. Moreover, the detection of time variation of fundamental constants [12] may be a strong evidence for the existence of extra-dimensions. Therefore, it is interesting to study string theory in higher dimension as both concepts are important at the early stages of the universe.

In this paper, we shall consider string theory in higher dimensional spherically symmetric space-time. The basic equations are constructed in § 2. Exact cosmological solutions of Einstein field equations are constructed in § 3 for different equations of state in string model. Also a polynomial relation between metric coefficients is assumed to solve the field equations. Section 4 deals with discussion of the results and conclusion therefrom.

2. The basic equations

We consider a spherically symmetric metric in 5-dimensional space-time with topology of 4-space $S^1 \times S^3$ as

$$dS^2 = -dt^2 + a^2 dr^2 + b^2 dQ_3^2, \tag{1}$$

where $a = a(t)$, $b = b(t)$ and

$$dQ_3^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2,$$

is the metric on unit 3-sphere. The co-ordinate r is periodic and varies in $[0, 2\pi]$.

The Einstein equation for a cloud of strings is

$$R_{\mu\gamma} - \tfrac{1}{2} g_{\mu\gamma} R = -(\rho \mu_\mu \mu_\gamma - \lambda x_\mu x_\gamma). \tag{2}$$

Here ρ is the rest energy density of the cloud of strings with particles attached to them (p-strings) and λ is the string's tension density. They are related by the relation [5,4]

$$\rho = \rho_p + \lambda, \tag{3}$$

with ρ_p being the particle energy density. u^μ is the five velocity for the cloud of particles and x^μ is the direction of anisotropy i.e. the string's direction, and they satisfy [4]

$$U_\mu U^\gamma = -1 = -x_\mu x^\gamma \quad \text{and} \quad U_\mu \cdot x^\gamma = 0, \tag{4}$$

in $(-, +, +, +, +)$ signature.

If we use co-moving co-ordinate system i.e. $U^\mu = U_\mu = (1, 0, 0, 0, 0)$ and x^μ parallel to $\partial/\partial r$ i.e. $x^\mu = (0, a^{-1}, 0, 0, 0)$, then the non-vanishing components of Einstein equation (2) for the metric (1) are

$$3\frac{b'^2}{b^2} + 3\frac{a'}{a}\frac{b'}{b} + \frac{3}{b^2} = \rho, \tag{5}$$

$$3\frac{b''}{b} + 3\frac{b'^2}{b^2} + \frac{3}{b^2} = \lambda, \tag{6}$$

$$\frac{a''}{a} + 2\frac{b''}{b} + \frac{b'^2}{b^2} + 2\frac{a'}{a}\frac{b'}{b} + \frac{1}{b^2} = 0. \tag{7}$$

The Bianchi identities reduce to

$$\rho' + \frac{a'}{a}(\rho - \lambda) + 3\frac{b'}{b}\rho = 0. \tag{8}$$

The proper volume R^4, expansion scalar θ and shear scalar σ are respectively given by

$$R^4 = a.b^3,$$

$$\theta = U^\mu_{;\mu} = \frac{a'}{a} + 3\frac{b'}{b},$$

$$\sigma^2 = \frac{a'^2}{a^2} + 3\frac{b'^2}{b^2} - \frac{1}{4}\theta^2. \tag{9}$$

The different equations of state for string model be [5].

(a) $\rho = \lambda$ (geometric string),

(b) $\rho = (1 + W)\lambda$ (Takabayasi string),

(c) $\rho = \rho(\lambda)$ (barotropic equation of state).

In the following section, we shall determine the exact solution of Einstein equations using above equations of state for string model.

3. Exact solution to Einstein equations

Here we have three field equations connecting four unknowns namely a, b, ρ and λ. So for unique solution one must assume one more relation connecting them.

Case I: Barotropic equation of state

In this case we assume polynomial relation between the two metric coefficients as [13]

$$a = \mu b^n, \tag{10}$$

where μ and n are constants.

In view of this equation (7) now becomes

$$(n+2)\frac{b''}{b} + (n^2+n+1)\frac{b'^2}{b^2} = -\frac{1}{b^2}.$$

For $n \neq -2$ ($n = -2$ implies b' imaginary) the above second order differential equation has a first integral of the form

$$b'^2 = A\cdot b^{-2(n^2+n+1)/n+2} - \frac{1}{n^2+n+1},$$

with A as the constant of integration. This differential equation can be written in the integral form as

$$\int \frac{\mathrm{d}b}{\left[A\cdot b^{-2(n^2+n+1)/n+2} - \dfrac{1}{n^2+n+1}\right]^{1/2}} = \pm(t - t_0), \tag{11}$$

where t_0 is another constant of integration. From the above integral equation b can be obtained in closed form only for $n = \pm 1$ and we obtain

$$b^2 = \frac{A}{B} - B(t-t_0)^2, \tag{12}$$

with $B = 1/3$ or 1 according as $n = +1$ or -1 respectively. The other parameters have the following expressions:

For $n = +1$,

$$a = \mu\sqrt{3A + \tfrac{1}{3}(t-t_0)^2}, \quad \rho = [9A - \tfrac{1}{3}(t-t_0)^2]/[3A - \tfrac{1}{3}(t-t_0)^2]^2,$$
$$\lambda = 2/[3A - \tfrac{1}{3}(t-t_0)^2], \quad \rho_P = [3A + \tfrac{1}{3}(t-t_0)^2]/[3A - \tfrac{1}{3}(t-t_0)^2]^2, \tag{13}$$

$$R^4 = \mu[3A - \tfrac{1}{3}(t - t_0)^2]^2, \quad \theta = \tfrac{4}{3}(t - t_0)/[3A - \tfrac{1}{3}(t - t_0)^2],$$
$$\sigma^2 = 0$$

For $n = -1$,

$$a = \mu/\sqrt{A - (t - t_0)^2}, \quad \rho = 3/[A - (t - t_0)]^2 = \rho_P,$$
$$\lambda = 0, \quad R^4 = \mu[A - (t - t_0)^2], \quad \theta = -2(t - t_0)/[A - (t - t_0)^2],$$
$$\sigma^2 = 3(t - t_0)^2/[A - (t - t_0)^2]^2. \tag{14}$$

Case II: *Geometric string* ($\rho = \lambda$)

In this case as $\rho_P = 0$, subtracting (5) from (6) we obtain

$a = \gamma b'$ (γ is an integration constant).

Now using the above expression for a in (7) the differential equation for b is

$$b^2 \frac{\mathrm{d}b''}{\mathrm{d}b} + 4bb'' + 1 + b'^2 = 0$$

which has a second integral

$b^2(1 + b'^2) = l\cdot b + m$, ($l, m$ are integration constants). This differential equation can be expressed in an integral form as

$$\int \frac{b\,\mathrm{d}b}{\sqrt{l\cdot b + m - b^2}} = \pm (t - t_0).$$

From this integral equation b can be expressed in closed form only for $l = 0$ and we have the following expressions for parameters:

$$\begin{aligned} b^2 &= m - (t - t_0)^2, \\ a &= \gamma(t_0 - t)/[m - (t - t_0)^2]^{1/2}, \\ \rho &= \lambda = 3[1 - m + (t - t_0)^2]/[m - (t - t_0)^2]^2, \\ \rho_P &= 0, \\ R^4 &= \gamma(t_0 - t)[m - (t - t_0)^2], \\ \theta &= [m - 3(t - t_0)^2]/(t - t_0)[m - (t - t_0)^2], \\ \sigma^2 &= \tfrac{3}{4}[m + (t - t_0)^2]^2/(t - t_0)^2[m - (t - t_0)^2]^2. \end{aligned} \tag{15}$$

For $l \neq 0$, the integral equation in b can be solved but b cannot be expressed explicitly as a function of t and consequently all physical parameters cannot be determined in terms of t. Therefore no physical conclusion can be drawn from the solution.

Case III: *Takabayasi string*

Here the equation of state is [14]

$$\rho = (1 + W)\lambda,$$

where $W>0$, a constant and it is small for string-dominated era and large for particle-dominated era. Now dividing (5) and (6) and using the above relation we have

$$(1+w)\frac{b''}{b}+w\frac{b'^2}{b^2}-\frac{a'}{a}\frac{b'}{b}+\frac{w}{b^2}=0.$$

Further using the polynomial relation $a=\mu b^n$ between the metric co-efficients the above relation and equation (7) become

$$(1+w)\frac{b''}{b}+(w-n)\frac{b'^2}{b^2}+\frac{w}{b^2}=0,$$

and

$$(n+2)\frac{b''}{b}+(n^2+n+1)\frac{b'^2}{b^2}+\frac{1}{b^2}=0.$$

Now eliminating b'' from the above relations we obtain

$$b'^2=k^2,\quad k^2=\frac{w(n+1)-1}{w(n^2-1)+(2n+1)(n+1)},$$

i.e.

$$b=k(t-t_0).$$

The other parameters are given by

$$a=\bar{\mu}(t-t_0)^n,\quad \bar{\mu}=\mu.K^n,$$

$$\rho=\frac{3}{K^2}[K^2(1+n)+1](t-t_0)^{-2},$$

$$R^4=\bar{\mu}K^3(t-t_0)^{n+3},$$

$$\theta=(n+3)(t-t_0)^{-1},$$

$$\sigma^2=\tfrac{3}{4}(n-1)^2(t-t_0)^{-2}. \tag{16}$$

4. Discussion and conclusion

In the above sets of solutions, except for p-string model all solutions show a contracting model of the universe and are not of much physical interest. The p-string solution will be interesting if $-3<n<0$. Then $t=t_0$ is the initial epoch of the universe. At this instant $b\to 0$, $a\to\infty$, $R^4\to 0$, $\theta\to\infty$, $\rho\to\infty$. So this is a line singularity. Thus the universe starts with an infinite rate of expansion and measure of anisotropy. Further, as t increases the scale factors a gradually decreases while the other one namely b increases. This can be interpreted as follows: If we identify the r co-ordinate as the extra dimension then it gradually decreases with the evolution of the universe i.e. the extra dimension becomes unobservably small as the universe evolves with time and we are left with the usual 4-dimensional space and topology of 3-space is S^3. Therefore, we may conclude that the extra dimension is important at the very early stages of evolution and then it gradually becomes unobservably small as expected.

References

[1] S Chakraborty, *Indian J. Pure Appl. Phys.* **29**, 31 (1991a)
[2] S Chakraborty, *Int. J. Theor. Phys.* **30**, 849 (1991b)
[3] S Chakraborty, *Astrophys. Space Sci.* **180**, 293 (1991c)
[4] A Banerjee, A K Sanyal and S Chakraborty, *Pramana – J. Phys.* **34**, 1 (1990)
[5] P S Letelier, *Phys. Rev.* **D28**, 2414 (1983)
[6] S Chatterjee, A Banerjee and B Bhui, *Phys. Lett.* **A149**, 91 (1990a)
[7] S Chatterjee and B Bhui, *Mon. Not. R. Astron. Soc.* **247**, 57 (1990b)
[8] T Fukui, *Gen. Relativ. Gravit.* **19**, 43 (1987)
[9] O Grøn, *Astron. Astrophys.* **193**, 1 (1988)
[10] E Witten, *Phys. Lett.* **B149**, 351 (1984)
[11] A Chodos and S Detweiler, *Phys. Rev.* **D21**, 2167 (1980)
[12] W J Marciano, *Phys. Rev. Lett.* **52**, 489 (1984)
[13] S Chakraborty and A Chakraborty, *J. Math. Phys.* (1992) (accepted)
[14] T Takabayashi, in *Quantum mechanical determinism, causality and particles*, edited by M Flato (D. Reidel Pub. Co. Holland, Dordrecht, 1978) p. 179