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A JWKB analysis of the sextic anharmonic oscillator in d dimensions

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Abstract. On the basis of a radial generalization of the JWKB quantization rule, which incorporates higher orders of the approximation, an explicit analytical formula is derived for the energy levels of the three-dimensional sextic anharmonic oscillator. The formula exhibits the scaling property of the exact eigenvalues, and is readily generalized to any dimension. The predicted results are in good agreement with known numerical values.

Keywords. Sextic anharmonic oscillator; JWKB formalism.

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1. Introduction

There exists quite substantial literature on the one-dimensional anharmonic oscillators (AHO's) with various types of anharmonicities. The AHO has in fact been used as a benchmark system to test the efficacy of any new or unconventional method proposed to tackle quantum mechanical and field theoretic problems. Relatively less attention has been bestowed on AHO's in higher dimensions with higher anharmonicities because of the greater complexity involved. The present work is devoted to the study of one such problem. One method which has proved useful in deriving analytical results for higher-dimensional AHO's is the JWKB approximation. While the lowest order JWKB formula for computing the energy levels is standard text book material, what is less known is the existence of an elegant analysis done long ago by Dunham [1] which enables one to compute higher order corrections systematically. With the corrections included, the JWKB formalism has led to explicit formulae for the energy levels of the pure AHO with the potential $V(r) = r^{2m}$ [2] and the quartic AHO with $V(r) = ar^2 + br^4$ [3], which reproduce the numerically computed eigenvalues quite well.

In the present work we apply the JWKB formalism to the 3-d AHO with a sextic anharmonicity, and obtain an analytical expression for its energy levels. We consider correction terms up to the fourth order in the quantization condition and compute them by the method of Krieger and Rosenzweig [4] which is an adaptation to radial problems of Dunham's pioneering one-dimensional analysis. Our analytical formula expresses the energy as a series in powers of the principal quantum number. The coefficients in this asymptotic series depend on the angular momentum l and the coupling constant. The first six of these are determined exactly. The scaling law obeyed by the exact eigenvalues of the problem is also obeyed by our expression for the energy. The predicted energy levels agree quite well with the numerical results available for this problem. It is a simple matter to extend our formula to the case of the sextic AHO in any higher dimension.

2. JWKB quantization of the sextic AHO

Consider a particle of unit mass moving in the potential

$$V(r)=\frac{1}{2}\alpha_1 r^2+\alpha_2 r^6, \quad \alpha_1,\alpha_2>0. \tag{1}$$

If $E_{nl}(\alpha_1,\alpha_2)$ denotes one of the allowed energy levels for a given angular momentum value l, then there exists an important identity

$$E_{nl}(\alpha_1,\alpha_2)=\alpha_2^{1/4}E_{nl}(\alpha_1\alpha_2^{-1/2},1). \tag{2}$$

This scaling property implies that it suffices to determine the eigenvalues $W(\lambda)\equiv E_{nl}(\lambda,1)$ of the Hamiltonian

$$H=\frac{1}{2}\mathbf{p}^2+\frac{1}{2}\lambda r^2+r^6 \tag{3}$$

corresponding to the potential $\tilde{V}=\frac{1}{2}\lambda r^2+r^6$. This we shall take as the standard form of the sextic AHO Hamiltonian.

To determine the eigenvalues W of the Hamiltonian (3), we employ the JWKB quantization condition

$$(2n_r+1)\pi=J_0+J_2+J_4 \tag{4}$$

which includes contributions from the second and fourth orders (J_2 and J_4 respectively) of the approximation. In (4) $n_r=0,1,2,\ldots$ is the radial quantum number. The JWKB integrals J_k are given by [4]

$$J_0=\sqrt{2}\oint\frac{\mathrm{d}r}{r}F^{1/2} \tag{5a}$$

$$J_2=-\frac{\sqrt{2}}{64}\oint\mathrm{d}r\,r\,F'^2F^{-5/2} \tag{5b}$$

$$J_4=-\frac{\sqrt{2}}{8192}\oint\mathrm{d}r\left[49r^3F'^4F^{-11/2}-16F'\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)^3F\cdot F^{-7/2}\right] \tag{5c}$$

with

$$F(r)=r^2\left(W-\frac{1}{2}\lambda r^2-r^6\right)-\frac{1}{2}\left(l+\frac{1}{2}\right)^2 \tag{6}$$

$$F'=\mathrm{d}F/\mathrm{d}r.$$

The above integrals are contour integrals in the r plane over a closed curve which surrounds a branch cut along the real axis between the two classical turning points defined to be the positive roots of $F(r)=0$. The contour does not enclose any other singularity of the integrand. The direction of integration depends on the branch of $F^{1/2}$ chosen. We take the contour to be traversed clockwise; it is then necessary to choose the branch of $F^{1/2}$ that is positive real on the upper lip of the branch cut.

It is an important feature of the JWKB formalism that the exact scaling relation (2) is also obeyed by the JWKB energy values determined by the quantization condition (4). Indeed, for polynomial potentials it can be shown that the exact scaling relation is satisfied by the JWKB energy values in every order of the approximation.

At this point we introduce for convenience the following new variable and parameters:

$$z = r^2 W^{-1/3}, \tag{7a}$$

$$\alpha = \frac{1}{8}\lambda W^{-2/3}, \tag{7b}$$

$$\sigma = \frac{1}{2}\left(l + \frac{1}{2}\right)^2 W^{-4/3}. \tag{7c}$$

Consequently,

$$F(r) = W^{4/3}(z - z^4 - 4\alpha z^2 - \sigma) \equiv W^{4/3} H(z). \tag{8}$$

In terms of z and H, we have

$$J_0 = \frac{1}{\sqrt{2}} W^{2/3} \oint \mathrm{d}z\, z^{-1} H^{1/2}, \tag{9}$$

$$J_2 = \frac{1}{6\sqrt{2}} W^{-2/3} \oint \mathrm{d}z (3z^3 + 2\alpha z) H^{-3/2}, \tag{10}$$

$$J_4 = -\frac{7}{8\sqrt{2}} W^{-2} \left[\frac{1}{3} \oint \mathrm{d}z (9z^7 + 12\alpha z^5 + 4\alpha^2 z^3) H^{-7/2} + \frac{3}{5} \oint \mathrm{d}z\, z^3 H^{-5/2} \right]. \tag{11}$$

In obtaining the above forms for J_2 and J_4 from (5) we have performed a number of integrations by parts.

We note that the leading W dependence of the integral J_k is $W^{2(1-k)/3}$. Apart from this explicit factor, the JWKB integrals depend on W only through the parameters α and σ, which go as $W^{-2/3}$ and $W^{-4/3}$, respectively. Consequent to the transformation $r \to z$, the integration in the z plane is around a branch cut joining z_1 and z_2, which are the real zeroes of $H(z)$. To first order in α and σ, $z_1 = \sigma$ and $z_2 = 1 - 4\alpha/3$.

3. Expansion of J_k in powers of α and σ

The quantization rule (4) with the integrals J_k given by (9) to (11) is a highly implicit relation for the allowed energy values. It may be expressed in terms of the three standard types of complete elliptic integrals whose arguments are complicated functions of W. For our present purpose however it is more advantageous to obtain series representations for the JWKB integrals directly in powers of W. The rationale of this procedure has been explained earlier [3]. Once the rhs of (4) is written as a series, it may be inverted to obtain W explicitly as a function of quantum numbers n_r and l, and the parameter α.

Since in the quantization condition we neglect terms J_k with $k \geqslant 6$, consistency requires that all terms proportional to $W^{-10/3}$ and lower powers be dropped from the series expansions of J_0, J_2 and J_4. Accordingly only terms up to and including $W^{-8/3}$ will be retained on the rhs of (4).

3.1 *Series for J_0:*

A careful inspection of J_0 shows that it can be expanded as a series in α and $\sqrt{\sigma}$. It turns out however that all terms involving odd powers of $\sqrt{\sigma}$ except the first vanish. Hence we can write

$$J_0 = \sum_{m,n=0} C_{mn}\alpha^m(\sqrt{\sigma})^n \tag{12}$$

where the coefficients are given by

$$C_{mn} = \frac{1}{m!\,n!}\left[\frac{\partial^{m+n}}{\partial\alpha^m\partial(\sqrt{\sigma})^n}J_0(\alpha,\sigma)\right]_{\alpha=\sigma=0}. \tag{13}$$

In using (13) we need to make use of the following result: If m, n are odd integers with $m \geqslant -1$, then

$$\oint \mathrm{d}z\, z^{m/2}(1-z^3)^{n/2} = \frac{2}{3}B\left(\frac{1}{3}+\frac{m}{6}, 1+\frac{n}{2}\right) \tag{14}$$

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where the contour of integration surrounds a branch cut from $z=0$ to 1 and B is the beta function defined by

$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y). \tag{15}$$

The result (14) is derived in the Appendix. Retaining terms up to $W^{-10/3}$, we can write down for J_0 the representation

$$J_0 = \frac{W^{2/3}}{\sqrt{2}}\left[-2\pi\sqrt{\sigma} + \sum_{k=0}^{5} a_k\alpha^k + \sigma\sum_{k=0}^{3} b_k\alpha^k + \sigma^2\sum_{k=0}^{1} c_k\alpha^k\right] + O(W^{-10/3}). \tag{16}$$

The coefficients in (16) are evaluated using (13) and (14). We get

$$a_0 = \frac{1}{2}B\left(\frac{1}{6},\frac{1}{2}\right) \equiv \frac{1}{2}B_1,\ a_1 = -\frac{4}{3}\pi,\ a_2 = \frac{8}{9}B\left(\frac{5}{6},\frac{1}{2}\right) \equiv \frac{8}{9}B_5,$$

$$a_3 = \frac{16}{81}B_1,\ a_4 = 0,\ a_5 = -\frac{640}{729}B_5,$$

$$b_0 = \frac{2}{3}B_5,\ b_1 = -\frac{4}{9}B_1,\ b_2 = 0,\ b_3 = \frac{320}{243}B_5,$$

$$c_0 = 0,\ c_1 = \frac{40}{27}B_5, \tag{17}$$

Using the properties of gamma function, the rhs of (14) in every case can be expressed as a rational number multiplied by either $B(\frac{1}{6},\frac{1}{2})$ or $B(\frac{3}{6},\frac{1}{2}) = \pi$ or $B(\frac{5}{6},\frac{1}{2})$.

3.2 *Series for* J_2:

The series representation for J_2 is similar to that of J_0, and somewhat simpler, since we need to keep terms only up to $\sigma\alpha$. An inspection of J_2 reveals that its expansion is free of all odd powers of $\sqrt{\sigma}$. Explicitly we have

$$J_2 = \frac{W^{-2/3}}{6\sqrt{2}}\left[\sum_{k=0}^{3} d_k\alpha^k + \sigma\sum_{k=0}^{1} e_k\alpha^k\right] + O(W^{-10/3}). \tag{18}$$

The coefficients are evaluated as earlier, and their values are

$$d_0 = -\frac{4}{3}B_5,\ d_1 = 0 = d_2,\ d_3 = \frac{1280}{243}B_5,$$

$$e_0 = 0,\ e_1 = -\frac{320}{27}B_5. \tag{19}$$

3.3 *Series for* J_4:

Referring to (11), we may write

$$J_4 = -\frac{7}{8\sqrt{2}}W^{-2}[f_0 + f_1\alpha] + O(W^{-10/3}) \tag{20}$$

up to the required power of W. Evaluating the coefficients we get

$$f_0 = 0,\ f_1 = 0. \tag{21}$$

Therefore the fourth order integral does not contribute any terms of order W^{-2} or $W^{-8/3}$.

4. Analytical formula for the energy levels

The series expansions derived above for the JWKB integrals lead to the following form for the quantization condition

$$\pi\sqrt{2}\left(n + \frac{3}{2}\right) = W^{2/3}\sum_{k=0}^{5} g_k W^{-2k/3} + O(W^{-10/3}) \tag{22}$$

where $n = 2n_r + l$ is the principal quantum number and we have used

$$\frac{1}{\sqrt{2}} W^{2/3} 2\pi\sqrt{\sigma} = \left(l + \frac{1}{2}\right)\pi: \tag{23}$$

The coefficients g_k are functions of λ and l, and are obtained by substituting for α and σ in terms of W, λ and l in the series expansions and regrouping terms according to powers of W. With the definitions

$$\beta = \frac{1}{8}\lambda,\ L^2 = \frac{1}{2}\left(l + \frac{1}{2}\right)^2 \tag{24}$$

the explicit expressions for the g_k are as follows;

$$g_0 = \frac{1}{2}B_1,\ g_1 = -\frac{4\pi}{3}\beta,\ g_2 = \frac{1}{9}(8\beta^2 + 6L^2 - 2)B_5,$$

$$g_3 = \frac{4}{9}\beta\left(\frac{4}{9}\beta^2 - L^2\right)B_1,\ g_4 = 0,$$

$$g_5 = \frac{640}{729}\left[\beta^3 - \beta^5 + \frac{3}{2}L^2\left(-\frac{3}{2}\beta + \beta^3\right) + \frac{27}{16}L^4\beta\right]B_5. \tag{25}$$

We note here an important feature of the relation (22). Since the leading term in J_6 is of the order of $W^{-10/3}$, the inclusion of J_6 and higher order integrals in (4) will not change the values of g_k in (25). Therefore these expressions for g_k are *exact to all orders.*

The relation (22), with the g_k given by (25) and (24), is the basis for an analytical formula for the energy levels to be obtained. If the leading term alone on the rhs of (22) is retained, the following expression is immediately obtained.

$$W = \left[\frac{2\sqrt{2}\pi}{B_1}\left(n + \frac{3}{2}\right)\right]^{3/2}. \tag{26}$$

This is the leading term in the asymptotic expansion of the energy of the sextic AHO (and also of the pure sextic oscillator $V(r) = r^6$). The inclusion of other terms in (22) results in corrections to this leading approximation. An inspection shows that when these corrections are included W is expressible as a series in powers of $(n + 3/2)$:

$$W = W_{nl} = N^{3/2} \sum_{k=0}^{5} G_k N^{-k} + O(N^{-9/2}) \tag{27}$$

where

$$N = \frac{2\sqrt{2}\pi}{B_1}\left(n + \frac{3}{2}\right). \tag{28}$$

The relation (27) represents the inversion of the series in (22). The coefficients G_k are functions of λ and l, and are determined by substituting (27) in (22). After a lengthy calculation we arrive at the following explicit expressions

$$G_0 = 1,\ G_1 = \frac{1}{2}a\lambda,\ G_2 = \frac{2}{3}b(1 - 3L^2) + \frac{\lambda^2}{24}(-b + a^2),$$

$$G_3 = \frac{\lambda}{18}[-2ab + 3(1 + 2ab)L^2] - \frac{\lambda^3}{864}(1 - 6ab + 2a^3),$$

$$G_4 = -\frac{2}{9}b^2(1 - 6L^2 + 9L^4) + \frac{\lambda^2}{36}[b^2 + a^2 b - 3(a + b^2 + a^2 b)L^2]$$

$$+ \frac{\lambda^4}{3456}(2a - 3b^2 - 6a^2 b + a^4),$$

$$G_5 = \frac{5}{27}\lambda[ab^2 + 3b(1 - 2ab)\, L^2 + 9ab^2\, L^4]$$

$$+ \frac{5}{1296}\lambda^3[-b(1 + 6ab + 2a^3) + 3a(3a + 6b^2 + 2a^2 b)\, L^2]$$

$$+ \frac{\lambda^5}{20736}[-5a^2 + 15ab^2 + 10a^3 b - a^5]. \tag{29}$$

In the above

$$L^2 = \frac{1}{2}\left(l + \frac{1}{2}\right)^2,$$

$$a = \frac{\pi}{B_1} = 0{\cdot}431 \quad 184 \quad 926 \quad 5,$$

$$b = \frac{B_5}{B_1} = 0{\cdot}307 \quad 509 \quad 933 \quad 9. \tag{30}$$

The relation (27), together with (28), (29) and (30) provides an explicit formula for the energy levels W_{nl} of the Hamiltonian (3). The eigenvalues E_{nl} of the general sextic AHO Hamiltonian (1) can be readily obtained from (27) through the scaling relation

$$E_{nl}(\alpha_1, \alpha_2) = \alpha_2^{1/4}\, W_{nl}(\lambda = \alpha_1 \alpha_2^{-1/2}). \tag{31}$$

Setting $\lambda = 0$ we obtain the special case of the pure sextic oscillator. The results coincide with those obtained earlier for $V(r) = r^{2m}$ when m is set equal to three [2].

It is a simple matter to derive from (27) a formula applicable to the sextic AHO in one dimension. Noting first that S waves of the three-dimensional problem correspond to the odd-parity levels of the symmetric one-dimensional potential of the same form, we can get the latter by setting $l = 0$ in (27) and replacing $n + 3/2$ by $n' + 1/2$ where n' is an odd integer. Since the coefficients G_k are independent of n, the same formula holds also for even values of n'. We thus obtain all the levels of the one-dimensional sextic AHO. For the general anharmonic oscillator in one dimension $[V(x) = \lambda x^2/2 + \mu x^{2k}]$ the first three terms in the asymptotic expansion of the energy in powers of $n + 1/2$ are known from the works of Hioe *et al* [5] and Pasupathy and Singh [6]. The one-dimensional version of our formula [27] is in complete agreement with these known results, and provides three more terms in the asymptotic expansion.

The energy level formula [27] can be readily generalized to the sextic AHO in any dimension $d > 1$, by making the replacements

$$l \to l + \frac{1}{2}(d - 3), \qquad n + \frac{3}{2} \to n + \frac{d}{3},$$

the second being a consequence of the first. The above replacement follows from the fact that the radial Schrödinger equation in d dimensions is [7]

$$\psi'' + 2\left\{E - V(r) - \frac{1}{r^2}\left[l + \frac{1}{2}(d-3)\right]\left[l + \frac{1}{2}(d-1)\right]\right\}\psi = 0.$$

A byproduct of our analytical formula is that the expectation values $\langle r^2 \rangle$ and $\langle r^6 \rangle$ in the (normalized) states of the sextic AHO can be calculated, using the Hellmann–Feynman (HF) theorem and the scaling relation (2). For the Hamiltonian (3) with eigenvalue $W(\lambda)$, we obtain

$$\langle r^2 \rangle = 2\frac{\partial W}{\partial \lambda}, \tag{32}$$

$$\langle r^6 \rangle = \frac{1}{4}(W - \lambda\langle r^2 \rangle). \tag{33}$$

Equation (33) is a special case of a general relation involving expectation values of powers of r for central potentials [8]. In the limit $\lambda \to 0$, (32) and (33) give the moments with respect to the pure sextic oscillator states. It is interesting that the calculation of the above two expectation values does not require a knowledge of the relevant wave function.

5. Results and discussion

In the tables we present the energy values obtained from our analytical expression for the sextic AHO in dimensions $d \leqslant 3$. We have chosen $\alpha_1 = 2\alpha_2 = 1$, as for this set of coupling strengths accurate numerically computed energy values are available [9, 10]. Tables 1–3 give in separate columns the energy levels computed in the leading order, second order and fourth order of the approximation. $E_{nl}^{(0)}$ is the lowest order value obtained by neglecting all terms beyond G_1 in our formula. $E_{nl}^{(2)}$ is the value incorporating second order corrections G_2 and G_3 while $E_{nl}^{(4)}$ corresponds to keeping all six coefficients G_0 to G_5. These numbers calculated from our formula are given up to the digit where they begin to deviate from the values in the last column. The

Table 1. Energy values of the three-dimensional sextic AHO with $H = (p^2 + r^2 + r^6)/2$.

n	l	$E_{nl}^{(0)}$	$E_{nl}^{(2)}$	$E_{nl}^{(4)}$	E_{nl}^{num}
0	0	0·24 E01	0·2506 E01	0·2507 E01	0·25166 97969 E01
1	1	0·40 E01	0·4734 E01	0·4729 E01	0·47277 67638 E01
3	1	0·144 E02	0·11254 E02	0·112521 E02	0·11251 96789 E02
3	3	"	0·102 E02	0·10176 E02	0·10169 30517 E02
5	1	0·195 E02	0·193535 E02	0·19352 72 E02	0·19352 69340 E02
5	3	"	0·1849 E02	0·18461 E02	0·18460 09648 E02
5	5	"	0·169 E02	0·16750 7 E02	0·16751 62768 E02
50	0	0·42060 E03	0·42061 2591 E03	0·42061 2591 E03	0·42061 25910 E03
50	50	"	0·338 E03	0·3298 E03	0·33084 61153 E03
500	0	0·12725 597 E05	0·12725 60192 E05	0·12725 60192 E05	0·12725 60192 E05
500	500	"	0·10 E05	0·984 E04	0·98743 09320 E04

Table 2. Energy values of the two-dimensional sextic AHO with $H = (p^2 + r^2 + r^6)/2$.

n	l	$E_{nl}^{(0)}$	$E_{nl}^{(2)}$	$E_{nl}^{(4)}$	E_{nl}^{num}
0	0	0·11 E01	0·155 E01	0·155 E01	0·15609 67737 E01
1	1	0·360 E01	0·3571 E01	0·3572 E01	0·35749 64301 E01
3	1	0·963 E01	0·95937 E01	0·959337 E01	0·95933 58711 E01
3	3	"	0·875 E01	0·8700 E01	0·86936 03904 E01
5	1	0·1734 E02	0·173089 E02	0·17308 705 E02	0·17308 70700 E02
5	3	"	0·1659 E02	0·165755 E02	0·16574 52759 E02
5	5	"	0·152 E02	0·15011 E02	0·15009 44032 E02
50	0	0·41451 E03	0·41453 0065 E03	0·41453 00645 E03	0·41453 00645 E03
50	50	"	0·333 E03	0·3251 E03	0·32610 27156 E03
500	0	0·12706 578 E05	0·12706 58429 E05	0·12706 58429 E05	0·12706 58429 E05
500	500	"	0·101 E05	0·982 E04	0·98595 66446 E04

Table 3. Energy values of the one-dimensional sextic AHO with $H = (p^2 + x^2 + x^6)/2$.

n	$E_n^{(2)}$	$E_n^{(2)}$	$E_n^{(4)}$	E_n^{num}
0	0·6 E00	0·75 E00	0·8 E00	0·71781 23095 E00
1	0·24 E01	0·2506 E01	0·2507 E01	0·25166 97969 E01
2	0·492 E01	0·49837 E01	0·49836 E01	0·49833 11000 E01
5	0·1527 E02	0·15311 E02	0·15311 302 E02	0·15311 29529 E02
10	0·3945 E02	0·39479 05 E02	0·39479 035 E02	0·39479 03429 E02
100	0·11438 87 E04	0·11438 96783 E04	0·11438 96783 E04	0·11438 96783 E04
1000	0·35850 027 E05	0·35850 03024 E05	0·35850 03025 E05	0·35850 03025 E05

last column in the tables gives the known numerically computed values ([9] for $d = 3$ and 2, [10] for $d = 1$). These values are accurate to 1 in 10^{15}, but we quote only ten significant figures as that is adequate for our purposes. We do not present results for other values of the coupling strengths because such accurate numerical results are not available for $d > 1$.

It is clear that the analytical formula reproduces the numerical results quite well, the agreement being strikingly good for large n, low l values, as it should be. Our results also bring out clearly the importance of including higher order corrections in the JWKB analysis. As the expansion parameter σ increases with l, the accuracy of the results is expected to decrease with increasing l for fixed n. This is evident from the tables. Nevertheless, even for the extreme case $n = l$, the predicted values are seen to be quite satisfactory. As expected, the ground state results are the poorest. Even here we may notice that the lowest order result can be improved by including higher order corrections, the accuracy increasing with the dimension. For very low values of n, the second order seems to yield a better result than the fourth, which is an indication that the series is asymptotic in nature.

We close with two remarks. The results for the harmonic oscillator cannot be recovered from our analytical formula. This is due to the fact that the expansion of the JWKB integrals is about the pure sextic oscillator levels ($\lambda = 0$). The harmonic oscillator results can of course be obtained from the integrals directly (in the limit

$\lambda \to \infty$) before they are expanded in series. Secondly, for given λ and classical energy E, the turning points in the sextic case are closer to each other than in the case of the quartic AHO. As the JWKB formalism will be more accurate the wider the separation between the turning points, it is to be expected that the JWKB results for the sextic AHO will be relatively less accurate than the corresponding quartic AHO results. This trend will be more pronounced for still higher anharmonicities.

Appendix. Proof of formula (14)

We consider here the evaluation of the integral

$$I(m,n) = \oint \mathrm{d}z\, z^{m/2}(1-z^3)^{n/2} \tag{A1}$$

where m, n are odd integers with $m \geqslant -1$, and the contour encircles a branch cut along the real axis from $z=0$ to 1. The square root is taken to be positive real on the upper lip of the cut, and the contour is traversed clockwise.

Taking first the case $n \geqslant -1$, we note that the integrand in (A1) has only integrable singularities at $z=0$ and $z=1$. The contour can therefore be deformed until it consists of two line segments lying just above and below the cut, together with two small circular arcs of radius ε around $z=0$ and $z=1$. In the limit $\varepsilon \to 0$, the contributions from the circular arcs vanish, and the integral reduces to

$$I = 2\int_0^1 \mathrm{d}x\, x^{m/2}(1-x^3)^{n/2}. \tag{A2}$$

By changing the variable to $t = x^3$ and using the definition

$$B(x,y) = \int_0^1 \mathrm{d}t\, t^{x-1}(1-t)^{y-1}$$

we get the result

$$I(m,n) = \frac{2}{3}B\left(\frac{1}{3}+\frac{m}{6}, 1+\frac{n}{2}\right). \tag{A3}$$

It now remains to prove that (A3) holds also when $n < -1$. Let $n = -(2k+1)$, $k = 1,2,3,\ldots$ To evaluate the integral, which is well defined as a contour integral, we consider

$$I(m,n,a) = \oint \mathrm{d}z\, z^{m/2}(a-z^3)^{-k-\frac{1}{2}}$$

and take the required integral to be $I(m,k,1)$. We can write

$$I(m,n,a) = C_k \frac{\partial^k}{\partial a^k}\oint \mathrm{d}z\, z^{m/2}(a-z^3)^{-1/2}$$

$$C_k = (-2)^k/(2k-1)!!$$

The a dependence of the integrand can be factored out by the change of variable

$y = a^{-1/3} z$. This yields

$$I(m,n,a) = C_k \frac{\partial^k}{\partial a^k} [a^{(m-1)/6}] \oint \mathrm{d}y\, y^{m/2} (1-y^3)^{-1/2}.$$

We observe that this integral is one with $n = -1$. Therefore its value can be found using (A3). Carrying out the differentiations and then setting $a = 1$, we get

$$I(m,n,1) = \frac{2}{3} C_k \frac{m-1}{6} \left(\frac{m-1}{6} - 1 \right) \cdots \left(\frac{m-1}{6} - k + 1 \right) B\left(\frac{1}{3} + \frac{m}{6}, \frac{1}{2} \right),$$

$$= \frac{2}{3} B\left(\frac{1}{3} + \frac{m}{6}, \frac{1}{2} - k \right) = \frac{2}{3} B\left(\frac{1}{3} + \frac{m}{6}, 1 + \frac{n}{2} \right),$$

the last step resulting from the use of the identity

$$B(x,y) = \frac{x+y}{y} B(x, y+1)$$

k times.

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