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Optical mode-coupling in a ring due to a back-scatterer

A VENUGOPALAN, DEEPAK KUMAR and R GHOSH
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110067, India

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Abstract. The coupling of light waves travelling clockwise and counterclockwise along an optical ring due to a back-scattering element is studied. An asymmetric mode splitting occurs as a consequence of the discontinuity suffered by the waves at the scattering point. The mode splitting shows up in an interference pattern and lends itself to an experimental verification.

Keywords. Optical mode-coupling; back-scattering in a ring; two-level system.

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1. Introduction

The analogy between quantum waves and classical waves has recently received a great deal of attention in connection with two-level systems in optics, both theoretically [1–4], and experimentally [5, 6]. In these studies optical ring cavities were used to implement two-level systems. The two levels were the two normal modes of the cavity created experimentally by lifting either the propagation or the polarization degeneracy of a single longitudinal mode of the ring. The two coupled modes constitute the model for phenomena such as photon-band structure in ring resonators [5, 6], frequency locking in laser gyros [7, 8] and statistical properties of bidirectional ring lasers [9–11]. In these studies the coupling of two modes of the electromagnetic field has been phenomenologically treated by regarding them as a pair of harmonic oscillators. While this model indeed contains the essential physics of two-level systems, it cannot be trusted to predict the variation of mode frequencies with the physical parameters which couple the initially degenerate modes. Further, in the case of perturbation by a back-scatterer the phenomenological model does not provide connection between the parameters of the model with the parameters of the back-scatterer. In this study we consider the effect of introducing a thin dielectric plate in a ring cavity and calculate explicitly the breaking of degeneracy between clockwise (CW) and counterclockwise (CCW) modes. It is shown here that this situation can again be reduced to a two-level system, but the parameters of the Hamiltonian depend, in a complicated way, on the parameters of the perturbation. The detailed variation of the frequency splitting is found to be different from what one obtains from the coupled oscillator model. We believe that the asymmetric mode-splitting predicted by this theory is fairly realistic, and can be subjected to experimental tests, particularly since all the parameters involved in the theory can be measured independently.

2. Theory of mode-coupling in a ring cavity

Consider the propagation of only one transverse mode in a one-dimensional optical ring cavity of length L. The dynamics can be described by a one-dimensional wave equation for $\phi(x,t)$ travelling along x-direction:

$$\left[\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right]\phi(x,t) = 0, \tag{1}$$

where c is the velocity of light. Since $\phi(x,t)$ has a two-fold propagation degeneracy, CW and CCW, it can be written as a two-component spinor, and the wave equation naturally decomposes to:

$$\begin{bmatrix} \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{bmatrix} = 0, \tag{2}$$

where $\phi_1(x,t)$ and $\phi_2(x,t)$ are the CW ($\sim \exp(ikx)$) and CCW ($\sim \exp(-ikx)$) propagating waves. The eigen-frequencies of the uncoupled CW and CCW waves are degenerate. In the ring, these lead to a standing wave pattern with $k = 2\pi n/L$ ($n = 0, 1, 2, \ldots$). This degeneracy can be lifted by introducing a coupling between the CW and CCW waves. If a back-scattering element is introduced in the cavity at point $x = 0$ (or $x = L$), the system can be described by the equation [12]:

$$\left\{\begin{bmatrix} \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \end{bmatrix} + i\delta(x)c\begin{bmatrix} A & B \\ B^* & A \end{bmatrix}\right\} \begin{bmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{bmatrix} = 0, \tag{3}$$

where A is real. The potential chosen here is hermitian and (3) satisfies the continuity equation with $c(\phi_1^*\phi_1 - \phi_2^*\phi_2)$ as the current density. The coefficients A and B (if real) can be related to the parameters of the back-scatterer by comparing the transmission (reflection) properties of this potential with those of a dielectric slab of thickness d and refractive index n_2 in the limits of plate thickness $d \to 0$ and the refractive index of the dielectric $n_2 \to \infty$ such that the product $n_2^2 d$ remains finite. The transmission and reflection coefficients of the potential are:

$$T = \left(1 + \frac{|B|^2 - A^2}{4} - iA\right)\Bigg/\left(1 + \frac{A^2 - |B|^2}{4}\right), \tag{4}$$

$$R = -iB\Bigg/\left(1 + \frac{A^2 - |B|^2}{4}\right). \tag{5}$$

The transmission and reflection coefficients of the slab are:

$$T' = \left(\cos(k_0 n_2 d) + i\frac{(n_1^2 + n_2^2)}{2n_1 n_2}\sin(k_0 n_2 d)\right)\exp(-in_1 k_0 d), \tag{6}$$

$$R' = i\frac{(n_2^2 - n_1^2)}{2n_1 n_2}\sin(k_0 n_2 \mathrm{d}). \tag{7}$$

The comparison in the limits mentioned above yields:

$$A = B = -(k_0 n_2^2 d)/n_1, \tag{8}$$

where k_0 is the wave vector in vacuum and n_1 is the refractive index of the medium in which the dielectric is embedded. In the absence of the dielectric, i.e., when $n_1 = n_2$ and there is no perturbation at $x = 0$, we have $A = 0 = B$. The scattering element and the cavity are assumed lossless [13]. The steady-state equations for the two counter-propagating waves with frequency W are:

$$i\frac{d}{dx}\phi_1 - \delta(x)(A\phi_1 + B\phi_2) = \frac{w}{c}\phi_1, \tag{9}$$

$$-i\frac{d}{dx}\phi_2 - \delta(x)(A\phi_2 + B^*\phi_1) = \frac{w}{c}\phi_2. \tag{10}$$

Integrating (9) and (10) across the potential, we get

$$i(\phi_1(L)) - \phi_1(0)) = \frac{A}{2}\left(\phi_1(0) + \phi_1(L)\right) + \frac{B}{2}\left(\phi_2(0) + \phi_2(L)\right), \tag{11}$$

$$-i(\phi_2(L) - \phi_2(0)) = \frac{A}{2}\left(\phi_2(0) + \phi_2(L)\right) + \frac{B^*}{2}\left(\phi_1(0) + \phi_1(L)\right). \tag{12}$$

The values of $\phi_1(x)$ and $\phi_2(x)$ at $x = 0$ (or $x = L$) have been taken as the averages $(\phi_1(0) + \phi_1(L))/2$ and $(\phi_2(0) + \phi_2(L))/2$. The scattering potential causes discontinuities in the wave functions [14] which means that $\phi_{1,2}(L) \neq \phi_{1,2}(0)$. Using

$$\phi_1(L) = \phi_1(0)\exp(ikL) \quad \text{for the CW wave,} \tag{13}$$

$$\phi_2(L) = \phi_2(0)\exp(-ikL) \quad \text{for the CCW wave,} \tag{14}$$

the eigenvalue equation for k assumes the form:

$$\begin{bmatrix} i(1-\exp(ikL)) + \frac{A}{2}(1+\exp(ikL)) & \frac{B}{2}(1+\exp(-ikL)) \\ -\frac{B^*}{2}(1+\exp(ikL)) & i(1-\exp(-ikL)) - \frac{A}{2}(1+\exp(-ikL)) \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} = 0. \tag{15}$$

The condition for nontrivial solutions of $\phi_1(0)$ and $\phi_2(0)$ yields two modes k_+ and k_- given by:

$$\tan(k_+ L/2) = (-A + |B|)/2, \tag{16}$$

$$\tan(k_- L/2) = -(A + |B|)/2. \tag{17}$$

Figure 1 shows the plot of the fundamental modes $k_\pm$ as the coupling $|B|$ is varied for a dielectric scatterer when (8) is obeyed. Note that for $A = |B|$, (16) gives the free cavity modes. For arbitrary A and B, the two solutions correspond to the two shifted modes of the cavity. The situation is analogous to a quantum two-level system, with the two modes $k_\pm$ representing a quantum two-level system with frequencies

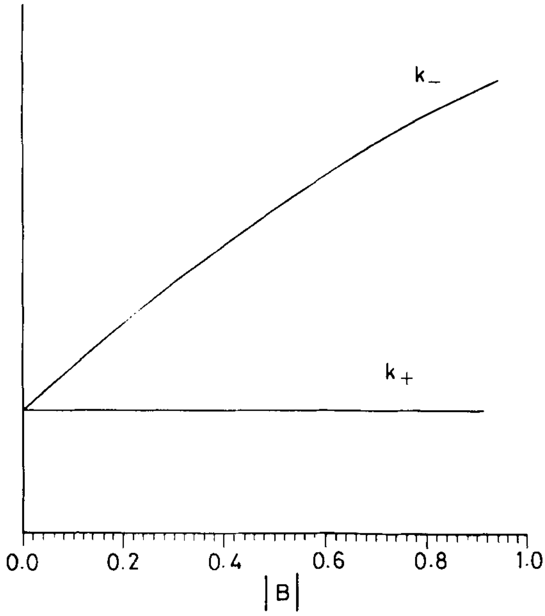

Figure 1. Plots of the fundamental modes $k_{\pm}$ versus the coupling parameter $|B|$ for a dielectric scatterer satisfying condition (8).

$w_{\pm} = ck_{\pm}$. The frequency difference between the two modes can be written as

$$\Delta w = (2c/L)\tan^{-1}\left(\frac{|B|}{1-(|B|^2 - A^2)}\right). \tag{18}$$

Note that this is different from the coupled harmonic oscillator model, where $2(\Delta^2 + W^2)^{1/2}$ is the difference in eigen-frequencies of the effective two-level Hamiltonian,

$$H = \begin{bmatrix} \Delta & W \\ W & -\Delta \end{bmatrix}, \tag{19}$$

in which Δ^2 is assumed to be proportional to the reflectivity, and the coupling rate W is taken to be constant. The difference in the qualitative predictions of (18) and (19) can, of course, be subjected to experimental test. It may be mentioned that by assigning complicated variations to both Δ and W, (18) can be recovered.

The amplitude ratios of the CW and CCW waves for each of our modes are

$$\frac{\phi_1^+(0)}{\phi_2^+(0)} = -\frac{B}{|B|}\frac{\left(1+i\dfrac{(A-|B|)}{2}\right)^2}{\left(1+\dfrac{(A-|B|)^2}{4}\right)} = \exp(i\gamma_+), \quad \text{for mode } k_+, \tag{20}$$

$$\frac{\phi_1^-(0)}{\phi_2^-(0)} = \frac{B}{|B|}\frac{\left(1+i\dfrac{(A+|B|)}{2}\right)^2}{\left(1+\dfrac{(A+|B|)^2}{4}\right)} = \exp(i\gamma_-), \quad \text{for mode } k_-. \tag{21}$$

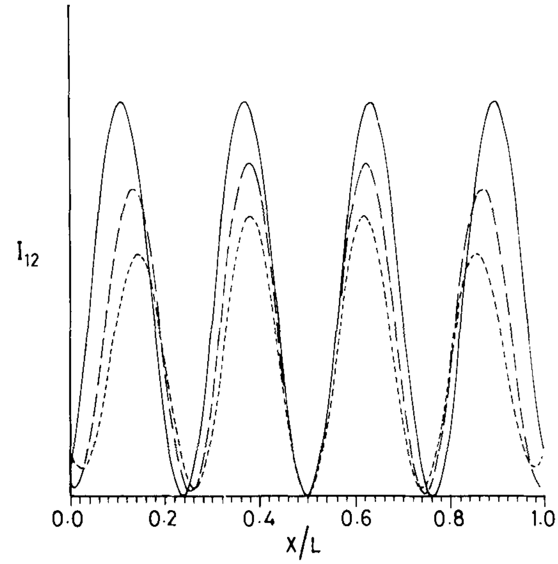

Figure 2. Plots of intensity I_{12} (in arbitrary units) versus relative position x/L at $t=0$ for a dielectric scatterer satisfying condition (8) with $|B|=0$ (———), $|B|=1$ (— — — —), $|B|=2$ (-----).

Note further that $\gamma_{\pm} = \pi - k_{\pm} L$, which implies

$$\phi^{\pm}(x) \propto \exp(-ik_{\pm}x) - \exp(ik_{\pm}(x-L)). \tag{22}$$

The general solution for the eigenmodes would contain components of both clockwise and counterclockwise waves in different proportions:

$$\phi^{+}(x,t) = \phi_1^{+}(0)\exp(ik_{+}(x-ct)) + \phi_2^{+}(0)\exp(-ik_{+}(x+ct)), \tag{23}$$

$$\phi^{-}(x,t) = \phi_1^{-}(0)\exp(ik_{-}(x-ct)) + \phi_2^{-}(0)\exp(-ik_{-}(x+ct)). \tag{24}$$

Due to relation (22) this implies

$$\phi^{\pm}(x,t) = \phi_2^{\pm}(0)[\exp(-ik_{\pm}x) - \exp(ik_{\pm}(x-L))]\exp(-ik_{\pm}ct). \tag{25}$$

The intensity of the superposed CW and CCW waves is given as

$$I_{12} = |\phi^{+}(x,t) + \phi^{-}(x,t)|^2. \tag{26}$$

Figure 2 shows the variation of I_{12} (in arbitrary units) along the ring at time $t=0$ for a dielectric scatterer satisfying (8), for two values of the coupling $|B|$. The modulation of the intensity pattern depends on the value of the coupling parameter B, which in turn determines k_{+} and k_{-}.

3. Summary

In summary, we have used a simple model of scattering of two optical modes by a partial reflector in a passive ring cavity to study the effect of conservative mode coupling. A modulated intensity pattern of the superposed CW and CCW waves indicates the mode structure, which can be tuned by the coupling parameter, B. This simple calculation gives explicit forms for the frequencies of the split modes in terms of the parameters of the back-scatterer. It can be seen that this frequency splitting cannot be easily reproduced in terms of the coupled oscillator model.

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[12] Equation (3) may be compared with the wave equation in an inhomogeneous linear medium in one dimension given by the Maxwell equations:

$$\frac{\partial^2 E}{\partial t^2} - v^2\frac{\partial^2 E}{\partial x^2} - v^2\frac{\partial}{\partial x}\left(\frac{1}{n^2}\frac{\partial n^2}{\partial x}E\right) = 0,$$

where E is the electric field, n is the refractive index of the medium, and $v = c/n = 1/n\sqrt{\mu_0 \varepsilon_0}$ is the velocity in the medium. In our model we have a local discontinuity of the refractive index of the medium at $x = 0$.
[13] To allow for injection of light, the cavity would necessarily be leaky and its resonances would have finite width. We assume here that losses affect all modes the same way and produce no coupling between [cf. [2]].
[14] The discontinuity of the amplitude $\phi = (\phi_1 + \phi_2)$ across the back-scatterer (arising from the delta function potential) may bother some readers, but it is seen quite physical when one realises that the electric field of the wave is discontinuous across the dielectric slab. It will be seen that due to relations (20) and (21) the amplitudes of the normal modes have the property $\phi^{\pm}(0) = -\phi^{\pm}(L)$.