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Inflationary solutions in general scalar tensor theory of gravitation

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Abstract. Extended inflation solution in Brans–Dicke theory given by Mathiazhagan and Johri (MJ) is shown as the unique solution only if the scale factor is assumed to be a power function of the scalar field. Only the consistent solution amongst the set of solutions given by Patra, Roy and Ray is found identical to the MJ solution. Both exponential inflation and power function inflation are studied in general scalar tensor theory where the parameter ω is a function of the scalar field. It is noted that exponential inflation is forbidden in Brans–Dicke theory where ω is a constant.

Keywords. Cosmology; inflation; scalar tensor theory.

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1. Introduction

Extended inflation given by La and others [1, 2] was based on the inflationary solutions obtained by Mathiazhagan and Johri [3] in Brans–Dicke theory of gravity [4]. Brans–Dicke theory with constant ω parameter is ruled out in view of the recent light deflection and time delay experiments [5] supporting values of $\omega > 500$. Such theories in fact lead to Einstein's theory of gravity in the limiting case of $\omega \rightarrow \alpha$. It is therefore worthwhile to consider a more general scalar tensor theory [6, 7, 8] where ω is a said to be a function of the scalar field φ and thus varies with time. It is worthwhile to mention here that limitations of extended inflation had been discussed by Weinberg [9].

In § 2 it has been shown that the well known solution of Mathiazhagan and Johri (hereinafter referred to as MJ) [3] is the unique solution for vacuum energy only when the scale factor $R(t)$ is assumed to be a power function of the scalar field φ. It is further shown that the other solutions in Brans–Dicke theory given by Patra *et al* [10] are not consistent with the field equations. One of them satisfies the field equations only in a special case but it is exactly the same as MJ solution and hence cannot be claimed to be a new solution.

In § 3 we have shown that the old inflationary solution given in the form of an exponential function of time is fully consistent with a variable ω but is not admissible in any case in the BD theory. We have given an exact solution of this type in the general scalar tensor theory where the function ω increases with time and may be quite large in the present era.

At the end we present another particular solution with the scalar factor R growing not exponentially but rather as power function of time. The parameter ω found in this case, however, does not enable us to draw any definite conclusion about its time behaviour throughout.

2. Field equations and discussion of solutions with constant ω

The general scalar tensor theory with $\omega = \omega(\varphi)$ was originally proposed by Bergmann [6], Wagoner [7] and Nordtvedt [8]. The action in that case is

$$S = \int d^4x\sqrt{-g}\left(-\varphi R - \frac{\omega}{\varphi} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 16\pi L_m\right) \tag{1}$$

where φ is the scalar field, ω is a parameter, being in general an arbitrary function of time and L_m includes a Higgs type sector which undergoes a strongly first order phase transition at high temperatures. The gravitational field equations corresponding to action (1), with the spatially flat ($k = 0$) Robertson–Walker metric, are

$$\frac{R'^2}{R^2} = \frac{8\pi\rho_v}{3\varphi} + \frac{\omega}{6}\frac{\varphi'^2}{\varphi^2} - \frac{R'}{R}\frac{\varphi'}{\varphi}, \tag{2}$$

$$2\frac{R''}{R} + \frac{R'^2}{R^2} = \frac{8\pi\rho_v}{\varphi} - \frac{\omega}{2}\left(\frac{\varphi'}{\varphi}\right)^2 - 2\frac{R'}{R}\frac{\varphi'}{\varphi} - \frac{\varphi''}{\varphi} \tag{3}$$

and the scalar wave equation for the scalar field is

$$\varphi'' + 3\frac{R'}{R}\varphi' = \frac{1}{(2\omega+3)}\left[32\pi\rho_v - \varphi'^2\frac{d\omega}{d\varphi}\right] \tag{4}$$

where prime represents differentiation wrt time. In the above, the false vacuum energy density ρ_v is given by $\rho_v = -p = \text{constant}$. Only two of the above equations are independent and there are three unknown functions of time such as R, φ and ω. It is therefore possible to make one assumption to get exact solutions. Combining (2) and (3) one has

$$2\frac{R''}{R} + 4\frac{R'^2}{R^2} = \frac{16\pi\rho_v}{\varphi} - 5\frac{R'}{R}\frac{\varphi'}{\varphi} - \frac{\varphi''}{\varphi}. \tag{5}$$

If we now assume $R^3 = \varphi^\alpha$, that is, the scale factor as a power function of φ, (5) on integration yields the first integral

$$\varphi'^2 = \frac{2C}{(2\alpha+1)}\varphi + D\varphi^{-2\alpha} \tag{6}$$

with the constant $C = 16\pi\rho_v/(2\alpha/3 + 1)(\alpha \neq -3/2)$ and D being an integration constant. The solution for the scalar field φ may be obtained after second integration, we have

$$t = \int \frac{d\varphi}{[(2c/(2\alpha+1))\varphi + D\varphi^{-2\alpha}]^{1/2}}. \tag{7}$$

Returning to the field equation (2) one has

$$\left(\frac{\alpha^2}{9} + \frac{\alpha}{3} - \frac{\omega}{6}\right)\frac{\varphi'^2}{\varphi} = \frac{8\pi\rho_v}{3} = \text{constant}. \tag{8}$$

In BD theory $\omega = \text{constant}$ which leads to $\varphi'^2/\varphi = \text{constant}$ and it follows therefore from (6) that $D = 0$. The exact solution for φ is therefore

$$\varphi = \left[\frac{1}{2}\left(\frac{2c}{2\alpha + 1}\right)^{1/2} t + t_0\right]^2 \tag{9}$$

It is possible to identify $\alpha = \frac{3}{4}(2\omega + 1)$ from (4) and (5) and one immediately identifies (9) with MJ solution. If D is a nonzero constant, one can get solutions with variable ω. Patra *et al* claim to have given the complete set of solutions in BD theory, but were not apparently careful about the field equations. They have given altogether five sets of solutions, the last two of which are simply first order differential equations in themselves. The first solutions is unphysical because the scalar field φ is negative. The second and third solutions are within the class of solution, where $R(t)$ is a power function of the scalar field and thus must yield the MJ solution. The field equation relates the constants A and B in the second solution such that it is exactly the MJ solution. But the third solution is inconsistent with the field equations and leads to absurd results.

3. Solutions with variable ω

It may be interesting to construct inflationary solutions in the general scalar tensor theory, where the parameter ω is small initially and subsequently grows with time. We find that although BD theory does not admit the exponential solution for $R(t)$ in the false vacuum stage, the more general scalar tensor theory has no such problem. We assume

$$R(t) = a\exp(ht) \tag{10}$$

a and h being constants. Using (10) in (5) one can write

$$\varphi'' + 5h\varphi' + 6h^2\left(\varphi - \frac{k}{6h^2}\right) = 0$$

so that the solution for φ is in general

$$\varphi = \frac{k}{6h^2} + A\exp(-2ht) + B\exp(-3ht)$$

A and B being two arbitrary constants. Making one of them, say B, to be zero for simplicity and noting that $A < 0$ is necessary to make $(2\omega + 3)$ positive, we can finally write the solution for φ in the form

$$\varphi = \left(\frac{k}{6h^2} - \beta^2\exp(-2ht)\right) \tag{11}$$

here $k = 16\pi\rho_v$ and $\beta^2 = -A = \text{constant}$.

The field equation (2) therefore yields $\omega(\varphi)$ in the form

$$\frac{1}{(2\omega + 3)} = \left(\frac{1}{3} - \frac{2h^2\varphi}{k}\right) = \frac{2\beta^2 h^2}{k}\exp(-2ht). \tag{12}$$

This relation is close to the choice of $\omega(\varphi)$ given by Bellido and Quirós [11]. Here

$(2\omega + 3) > 0$, and is thus consistent with the demand of the theory. Both φ and ω increase with time. It is evident from (12) that no meaningful solution for $R(t)$ as an exponential function of time exists when $\omega = \text{constant}$, that is in Brans–Dicke theory. It therefore follows that inflation of exponential nature is not consistent with BD theory.

On the other hand, one may attempt to obtain the scale factor $R(t)$ as a power function of t, that is $R^3(t) = at^n$, where n and a are constants. Using (5) with the assumed expression for the scale factor one gets,

$$\varphi'' + \frac{5}{3}n\frac{\varphi'}{t} + \frac{2n(n-1)}{3}\frac{\varphi}{t^2} = 16\pi\rho_v. \tag{13}$$

If $n = 6$, equation (13) can be readily integrated to obtain

$$\varphi = \frac{8\pi\rho_v}{21}t^2 + a_1 t^{-4} + a_2 t^{-5}, \tag{14}$$

a_1 and a_2 being constants of integration. When $a_1 = a_2 = 0$, the above solution for φ reduces to that in BD theory, that is the MJ solution, for a particular choice of ω. For varying ω, we choose $a_2 = 0$ for simplicity so that the expression for ω is

$$\omega = 24\left(\frac{16\pi\rho_v}{21}t - 4a_1 t^{-5}\right)^{-2}\left(\frac{8\pi\rho_v}{21}t + a_1 t^{-5}\right)\left(\frac{2\pi\rho_v}{21}t - a_1 t^{-5}\right). \tag{15}$$

Study of the solution (14) for φ reveals that (for $a_2 = 0$) at $t \to 0$, $\varphi \to \infty$ provided $a_1 > 0$ and also φ increases indefinitely in the limit $t \to \infty$. So φ has a minimum at some finite time. In this case, $(2\omega + 3)$ remains positive for $t > (21a_1/2\pi\rho_v)^{1/6}$ and as $t \to \alpha$, $(2\omega + 3) \to 6$.

It would have been interesting if solutions of (13) for other values of n were obtained. But we could not obtain analytic solutions for $n \neq 6$.

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