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# State reduction theories, their classical analogs and some comments

A M JAYANNAVAR
Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India

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**Abstract.** Recently several theories have been proposed to account for the state reduction due to measurement. The resulting evolution is given by a new density matrix equation which suppresses linear superpositions of states with large spatial separations. We raise some pertinent questions regarding these theories. We also show that the evolution for the density matrix obtained in these theories has a classical analog.

**Keywords.** State reduction; density matrix; pointer variable; white noise.

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## 1. Introduction

Quantum mechanics has so far been in perfect agreement with all experiments, yet the debate continues with regard to its interpretation. Quantum mechanics contradicts our general view of macroscopic objects. The linearity of the Schrödinger equation leads to superpositions of macroscopically distinguishable states and hence Schrödinger's cat paradox. Particularly in quantum mechanics, if we describe an initial state of a composite system in terms of non-factorized state vectors, for later times, state of a system continues to be in a non-factorized state even while their constituents (may be macroscopic in size) are far off in spatial domain and non-interacting. This fact forbids us from attributing a physical individuality to the components of the composite system (quantum non-separability). On the other hand we have the well tested theory namely, classical mechanics which describes the behaviour of the macroscopic composite objects with well defined individual physical attributes. Now the question arises as to whether we can obtain classical attributes from the quantum theory. To answer this question specifically, several state reduction theories have been developed. One class of theories treats the macroscopic body as a complex one interacting with an inevitable environment [1]. In fact one can show that the environment surrounding a quantum system induces decoherence leading to a classical behaviour [1, 2]. For macroscopic objects, quantum coherence would be destroyed in a very short time scale. These theories emphasize the fact that the loss of coherence can be regarded as an act of measurement [1]. If the quantum system is coupled to an environment through a dynamical variable of the system say $A$, then the variable $A$ is called a "pointer" variable [3, 4]. Pointer variable can also be defined as a dynamical variable $A$ of the system which commutes with the interaction Hamiltonian. This guarantees that the pointer observable is a constant of motion of the interaction Hamiltonian. Naturally, if the system is in an eigen state of $A$, the interaction with an environment will leave it unperturbed. The environment selects the preferred basis (pointer basis),

i.e., an environment acts as if it is measuring a dynamical variable $A$ of the system. Consequently the density matrix of the system reduces rapidly to the diagonal form (mixed state) in the pointer basis representation. The rate of the reduction depends on the strength of the coupling with the environment. This loss of coherence destroys the superpositions and hence gives rise to classical behaviour. One can now understand easily the nature of the system being quantum or classical, by its ability to retain quantum coherence. However, the procedure we have mentioned above subsumes the system and the environment to form a closed system. The evolution of the system plus environment is a Hamiltonian evolution and we look for the projected evolution of the system alone. This is true when we prescribe the total Hamiltonian, i.e. system, environment and interaction. One can also take a practical view that the measurement is a complex process and is an external act. Since we do not know the total Hamiltonian precisely it is difficult to formulate the reduced dynamics of a system through a Hamiltonian evolution.

The second class of theories takes a viewpoint that the system evolves deterministically according to the Schrödinger equation. When the quantum system interacts with an apparatus, set up for measuring a dynamical variable of the system, the state of the system changes abruptly after the measurement. The measurement collapses the state of the system into an eigen state of the measured dynamical variable. This second process cannot be treated in the framework of the Schrödinger equation. In order to reconcile with these two different types of evolutions and to suppress the unwanted linear superpositions of states far apart in spatial regions, Ghiradi *et al* have introduced quantum mechanics with spontaneous localization [5]. In this treatment, one invokes the position measurement, the particle besides evolving through a Hamiltonian dynamics, is subjected to repeated collapse corresponding to localization in space at random times. For a single particle in one-dimension the instantaneous localization process corresponds to

$$\psi_Q \rightarrow \exp[-\tfrac{1}{2}\alpha(\hat{x}-\bar{x})^2]\psi_Q, \tag{1}$$

which occurs at random interval of time. In each localization process the co-ordinate $\bar{x}$ is a random variable with probability distribution given by

$$P(\bar{x}) \alpha \langle \exp[-\alpha(\hat{x}-\bar{x})^2] \rangle, \tag{2}$$

where $\langle \ldots \rangle$ stands for quantum expectation value in the actual state $\psi_Q$, and $\alpha$ is the accuracy of the localization process. In every process of measurement when one collapses the wave function around the mean value of co-ordinate observable in quantum state $\psi_Q$, one expects this procedure will lead to a classical behaviour in terms of trajectories. Now due to the randomness in the localization process, the evolution of $\psi_Q$ is governed by a stochastic equation. Finally one arrives at the reduced equation for the density matrix $\rho(x, x^1, t)(\equiv \psi^*(x,t)\psi(x',t))$, namely,

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m}\left(\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x'^2}\right)\rho - \lambda f(x,x')\rho, \ldots \tag{3}$$

where $f(x,x')$ is given by $\{1-\exp[-(\alpha/4)(x-x')^2\}$. Diosi [6, 7] obtains the same equation (1), by taking into account the modification of quantum dynamics by gravitational effects. In his case $f(x,x') = (\gamma/4)(x-x')^2$. Joos and Zeh [8] have also arrived at the same master equation for a system interacting with an external environment.

Using (3) one can readily verify that $(d/dt)(tr\rho^2) < 1$. This implies that under dynamical evolution pure states are transformed into statistical mixture. These theories have been widely discussed in literature and in particular Bell [9] has made a clear exposition of it.

We show below that (3) has a classical analog. Moreover, if a quantum particle is subjected to a purely classical white noise potential, then the resulting density matrix evolution is given precisely by (3). We also point out certain inconsistencies regarding the very notion of the position measurement. Most of the presentation here is based on our earlier work related to quantum transport problems in a dynamically disordered medium [10].

## 2. Theory

Consider first a classical particle subjected to a random force,

$$m\frac{d^2x}{dt^2} = f(t), \tag{4}$$

where $f(t)$ is a Gaussian white noise. One can easily verify that [10] the mean-squared displacement of the particle $\langle x^2(t)\rangle$ grows as $t^3$. This is because the particle continues to absorb energy from the fluctuating force and accelerates indefinitely. Equation (4) represents the well-known Langevin equation in the absence of dissipative or frictional force [11]. Here we have not provided any dissipative mechanism, which is a must for a system to equilibrate or to reach a steady state. In short the particle heats up to infinite temperature. Now corresponding to the above classical equation the Hamiltonian is given by $H = p^2/2m - xf(t)$. Quantizing this system, the quantum Hamiltonian is given by

$$H = \frac{-\hbar^2}{2m}\nabla^2 - xf(t) \tag{5}$$

where $f(t)$ is a white noice force with $\langle f(t)\rangle = 0$, $\langle f(t)f(t')\rangle = 2V_0^2\delta(t-t')$. The angular brackets represent averaging over all realizations of random force. Using Schrödinger equation one can easily write down the equation for the averaged density matrix $\rho(x,x',t)\{\equiv \psi^*(x,t)\psi(x',t)\}$. Following exactly the same procedure due to [10] we arrive at the same master equation (1), where $f(x,x') = (V_0^2/\hbar^2)(x-x')^2$. This is exactly the equation derived earlier by Diosi [6, 7] with a redefinition of the coefficient, namely $\lambda\gamma/4 = V_0^2/\hbar^2$. The potential energy in (5) is given by $V(x,t) = -xf(t)$, if we replace this potential by a space time dependent Gaussian noise with statistics given by $\langle V(x,t)\rangle = 0$ and $\langle V(x,t)\, V(x',t')\rangle = g(x-x')\delta(t-t')$, we again arrive at (1) for the averaged density matrix (see equation (8) in [10]). If we assume spatial part of the correlation function to be $g(x-x') = \exp((-\alpha/4)(x-x')^2)$, we get the result derived by Ghirardi *et al* [5]. Thus we have shown that (3) has a classical analog and is precisely the density matrix equation obtained for a quantum particle subjected to classical white noise potential.

Now we will consider the question regarding the position measurement. As mentioned earlier theoretical schemes leading to (3) explicitly invoke the position measurement. Hence naturally one expects the evolution of the wave packet to be blocked by repeated position measurement. In quantitative terms one expects the slower growth of width (spread) of the wave packet as compared to the unperturbed free

evolution (in the absence of a measurement). In fact it is now very well-known that in the extreme case, a continuous position measurement completely inhibits the time evolution (quantum zero effect of watched pot effect) [12]. Such a physical effect also manifests as a reduction in the tunnelling rate due to coordinate coupling to the environment [13]. In this situation the co-ordinate coupling to the environment induces effectively a position measurement of a system, thus the width of a wavepacket evolves slowly thereby suppressing the tunnelling rate. In contrast to such a physical expectation, state reduction theories based on position measurement lead to faster evolution for the width (or spread) of the wave packet as compared to the free evolution. Consider a simple case, where $f(x,x') = (V_0^2/\hbar^2)(x-x')^2$ [6, 7]. We now take initial condition that the particle was prepared initially in a wave packet centred at origin $x=0$, i.e. $\psi(x,t=0) = (1/2\pi^4)(1/\sigma^{1/2})\exp[-x^2/4\sigma^2]$. This ensures correct normalization for the initial density matrix $\rho(x,x',t=0) = \psi(x,t=0)\psi^*(x,t=0)$. The initial wave function has a width (or spatial spread) characterized by $\langle x^2\rangle$ equal to $\sigma^2$. Following [10] one can evaluate exactly, the time development of the width or the mean-squared displacement and is given by

$$\langle x^2(t)\rangle = \sigma^2 + \frac{\hbar^2 t^2}{4m\sigma^2}\frac{V_0^2}{3m^2}t^3,$$

$$= \langle x^2(t)\rangle_f + \frac{V_0^2}{3m^2}t^3. \tag{6}$$

From (6) one can notice that the width of the wave packet spreads at a much faster rate when compared to the free-evolution $\langle x^2(t)\rangle_f$ for all times $t>0$. Hence we conclude that even though state reduction theories leading to (3) are based on position measurement, they give inconsistent result, i.e. they do not predict blocking of the width (spread) of the wave function, in fact the contrary is true. If one takes $f(x,x')$ given by [5], the same behaviour for $\langle x^2(t)\rangle$ is observed (see equation (17) in [10]). As shown earlier $t^3$ dependence of $\langle x^2(t)\rangle$ observed in these theories is contained in a classical analog. Also one can see clearly the objection by Ballantine [14] that these theories do not possess a steady or equilibrium state and the energy conservation is contained in the classical counterpart itself so has nothing to do with the quantum mechanics.

We now consider another interesting case when the spatial correlation of a random potential is given by $g(x-x') = \exp[-\alpha(x-x')^4]$. Such a correlation function conserves the energy on the average [14], indicating that $\langle x^2(t)\rangle = (\langle x^2(t)\rangle_f)$. However, $\langle x^4(t)\rangle \sim t^5$, indicates that the spread of the wave packet is faster.

## 3. Conclusions

The theories [5, 6, 7] for position measurement are probably good ones for describing the quantum motion of an accelerating particle interacting with a parametric random force (but not for the state reduction due to position measurement). Moreover we have explicitly shown that the problem of non-equilibration and steady state is present in the classical analog.

Finally we would like to point out that the scheme due to Ghirardi *et al*, namely the quantum evolution blocked by repeated collapse, is quite appealing. However, the problem is that these treatments are done in a continuum space. In such a situation the energy operator of a particle is unbounded and hence the particle continually

gains energy without any limit. If indeed one can carry out the same scheme [15] for a motion of a particle on a one-band lattice Hamiltonian (where the energy operator is bounded), one can readily obtain (see references in [10]) the result for the width of the wave packet as $\langle x^2(t) \rangle \sim t$. This is a much slower evolution as compared to the free evolution. This is consistent with the notion of the position measurement. Such treatments have been successfully used earlier to explain many physical phenomena, which include, quantum diffusion on lattices, spontaneous symmetry breaking and blocking of metastable states [16] and some transport related results in high temperature superconductivity [17].

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