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Chaos and curvature in a quartic Hamiltonian system

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Abstract. Chaotic behaviour of a quartic oscillator system given by $H = 1/2(p_1^2 + p_2^2) + (1/12)(1-\alpha)(q_1^4 + q_2^4) + 1/2q_1^2q_2^2$ is studied. Though the Riemannian curvature is positive the system is nonintegrable except when $\alpha = 0$. Calculation of maximal Lyapunov exponents indicates a direct correlation between chaos and negative curvature of the potential boundary.

Keywords. Chaos; Riemannian curvature; Lyapunov exponents.

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1. Introduction

Deterministic chaos in nonlinear dynamical systems has received much attention recently. Chaos is characterized by the exponential divergence of initially close trajectories as the system evolves. Such systems exhibit a sensitive dependence on initial conditions. To study chaos there exist different techniques mostly numerical. There are also nonlinear dynamical systems which are integrable and show ordered phase space motion. But there are no general methods to differentiate between integrable and nonintegrable ones. How and why chaos arises in some systems is an interesting question yet to be answered completely.

In this paper we study a quartic Hamiltonian system with two degrees of freedom and explore the connection between the chaotic behaviour and the Riemannian curvature of the manifold in which the Hamiltonian flow can be considered as a geodesic flow. We find that there is a direct link between the chaotic behaviour as measured by Lyapunov exponents and the negative curvature of the potential boundary. In § 2 we briefly describe the system under study and in § 3 details of the calculation of Lyapunov exponents are given. Relationship of dynamics with Riemannian geometry and Riemannian curvature of the system is given in § 4. In § 5 potential boundary curvature is calculated and § 6 contains our conclusions.

2. The Hamiltonian system

The Hamiltonian of the system is given by

$$H = 1/2(p_1^2 + p_2^2) + V(\mathbf{q}), \tag{1}$$

where

$$V(\mathbf{q}) = \frac{(1-\alpha)}{12}(q_1^4 + q_2^4) + 1/2q_1^2q_2^2 \tag{2}$$

and α is a parameter, $0 \leqslant \alpha \leqslant 1$. Potential $V(\mathbf{q})$ for different α values are plotted in

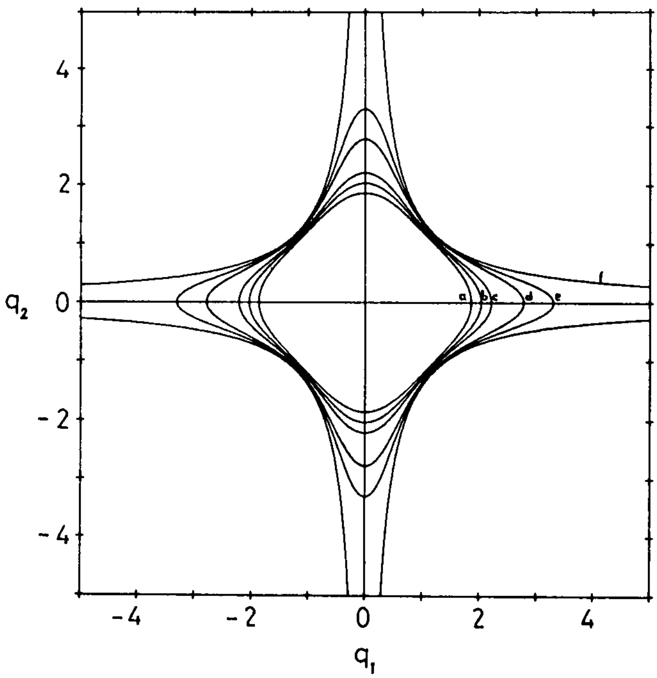

Figure 1. Curve of potential $V = 1$ for α values equal to (a) 0·0, (b) 0·3, (c) 0·5, (d) 0·8, (e) 0·9 and (f) 1·0.

figure 1, for $V = 1$. Chaos of this system has been studied in detail by Carnegie and Percival [1] using the techniques of Poincare surface of section and by studying the properties of periodic orbits. The system has got $\pi/4$ symmetry. At $\alpha = 0$ it is integrable and the corresponding second integral of motion is given by

$$I = 3p_1p_2 + q_1q_2(q_1^2 + q_2^2). \tag{3}$$

Phase space motion is regular and all trajectories lie on an invariant tori. As α increases regular regions break up and irregular regions appear. When $\alpha = 1$, system become highly chaotic which was shown to be equivalent to a K-system by Savvidy [2]. It has been used as a simplified model of the spatially homogeneous classical Yang-Mills field [2].

The system (1) is scale invariant and we can study the chaotic behaviour at a fixed value of energy $H = E$. By scaling we may obtain the behaviour at any other energy value. Using singular point analysis, Steeb *et al* [3] have shown that the system is nonintegrable except when $\alpha = 0$. The resonances or Kowalevskaya exponents are $r_1 = -1, r_2 = 4, r_{3,4} = 3/2 \pm (9/4 + 4(\alpha + 2)/(\alpha - 4))^{1/2}$. When $\alpha > 4/25$ resonances are complex. Integrability of such systems in general has been studied [4] using singular point analysis and stability analysis. Quantum chaos of this system has been studied [5, 6]. Effect of quantum fluctuations on this system has been investigated by us [7] by calculating the Gaussian effective potential.

3. Lyapunov exponents

We shall investigate the possibility of chaotic behaviour of the system by computing the maximal Lyapunov exponent (LE). LE is the average rate of exponential divergence of nearby trajectories [8]. If it is positive the system is said to be chaotic. Let the dynamical system be given by

$$x_i' = F_i(x), \tag{4}$$

$i = 1,\ldots,n$, where n is the dimension of the system. For calculating the LE we have to numerically solve the system (4) along with the corresponding variational system given by,

$$y_i' = (\partial F_i/\partial x_j)y_j, \quad i,j = 1,\ldots,n. \tag{5}$$

This gives us the evolution of the variation y of a nearby orbit. One-dimensional LEs are given by,

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\left|\frac{y(0)}{y(t)}\right|, \tag{6}$$

where $|y(t)|$ is the norm. For an n dimensional system there are n one-dimensional LEs. If we order them as $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$, then λ_1 is known as the maximal LE. For a Hamiltonian system 1-dimensional LEs are symmetric about zero and there will be at least two zero LEs. Sum of LEs will be zero. Hence for a chaotic Hamiltonian system with two degrees of freedom there will be only one positive LE. If we choose the initial variations at random we obtain maximal LE, with probability one. In numerical calculations we have to normalize the variation at appropriate time steps, to avoid overflow.

Equations of motion of our system (1) can be written as,

$$x_1' = x_3,\ x_2' = x_4,\ x_3' = (\alpha - 1)x_1^3/3 - x_1x_2^2,\ x_4' = (\alpha - 1)x_2^3/3 - x_1^2x_2. \tag{7}$$

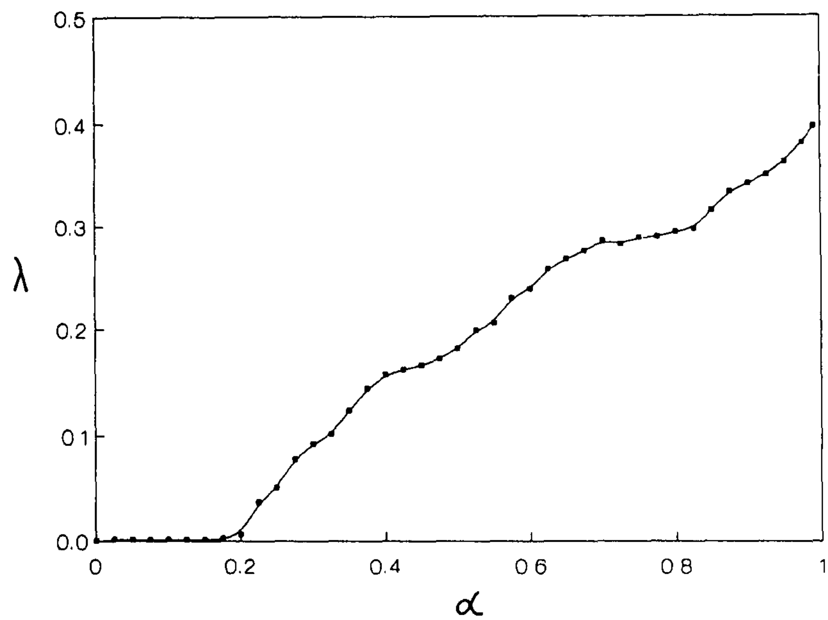

Figure 2. Plot of maximal LE (λ) vs α.

Corresponding variational system is given by

$$y_1' = y_3,\ y_2' = y_4,\ y_3' = ((\alpha - 1)x_1^2 - x_2^2)y_1 - 2x_1x_2y_2,$$
$$x_4' = ((\alpha - 1)x_2^2 - x_1^2)y_2 - 2x_1x_2y_1. \tag{8}$$

We numerically solve the systems (7) and (8) together. LE is calculated for different values of the parameter α, with different sets of initial variations. In figure 2 maximal LE vs α is plotted. We take the energy $E = 1$ for our calculations. As α increases, one can see from LE value that the chaos in the system also increases.

4. Riemannian curvature

Any Hamiltonian flow can locally be considered as a geodesic flow on a Riemannian manifold [9]. Consider a system with Hamiltonian of the form

$$H(\mathbf{p}, \mathbf{q}) = 1/2 \sum_{i,j=1}^{N} a_{ij}(\mathbf{q}) q_i' q_j' + V(\mathbf{q}). \tag{9}$$

The solutions $q_i(t)$ are extremes of Euler-Maupertuis principle,

$$\delta \int_{M_0}^{M_1} \left(2\{E - V(\mathbf{q})\} \sum_{i,j=1}^{N} a_{ij}(\mathbf{q}) \mathrm{d}q_i \mathrm{d}q_j \right)^{1/2} = 0 \tag{10}$$

where M_0 and M_1 are the end points of the trajectory. This may be considered as the variational equation for geodesics in a space with a line element.

$$\mathrm{d}s^2 = \sum_{i,j=1}^{N} g_{ij} \mathrm{d}q_i \mathrm{d}q_j \tag{11}$$

and metric coefficients

$$g_{ij} = 2\{E - V(\mathbf{q})\} a_{ij}(\mathbf{q}). \tag{12}$$

Evolution of the separation ρ between the nearby geodesics obey (to the lowest order in ρ) the Jacobi equation

$$\frac{D^2 \rho}{\mathrm{d}t^2} = -K(q, t)\rho, \tag{13}$$

where $D/\mathrm{d}t$ is the covariant derivative. If we restrict ourselves to initial separations perpendicular to an orbit, the covariant derivative can be replaced by ordinary one and we can write the Jacobi equation (13) as

$$\frac{d^2 \rho}{\mathrm{d}t^2} = -K(q, t)\rho, \tag{14}$$

where $K(q, t)$ is the Riemannian curvature calculated along the orbit.

When $E - V = 1/2 \Sigma P_i^2$, so that $a_{ij} = \delta_{ij}$, Riemannian curvature $K(q, t)$ is given by [10],

$$K = \frac{(N - 1)}{8(E - V)^3} \{2\,\mathrm{Tr}(\mu_{ij}) - N\, V_m\, V_m\} \tag{15}$$

where

$$\mu_{ij} = 3V_i V_j + 2(E - V) V_{ij},$$
$$V_i = \partial V/\partial q^i \text{ and } V_{ij} = \partial^2 V/\partial q^i \partial q^j.$$

In two dimensions K is same as Gaussian curvature. Sign of K indicates the stability of the orbit. Positive curvature implies local stability, whereas negative curvature means instability. Hadamard-Lobachevsky theorem suggests that if the Riemannian curvature is negative, the system behaves chaotically; there is exponential divergence of nearby trajectories. Surfaces of constant negative curvature are chaotic. Negative R-curvature everywhere is a sufficient condition for chaotic behaviour. But converse is not true. Positive curvature does not mean that the system is integrable. Local instability everywhere implies global instability but local stability everywhere does not imply global stability [11]. Association of the system with the geodesic flow on R-manifold is not valid at the boundary of the manifold, i.e., at $E = V$, where the metric tensor becomes singular.

In two dimensions R-curvature (Gaussian curvature) is given by,

$$K = 1/2(E - V)^2\{\partial^2 V/\partial q_1^2 + \partial^2 V/\partial q_2^2 + 1/(E - V)[(\partial V/\partial q_1)^2 + (\partial V/\partial q_2)^2]\},\ E > V. \tag{16}$$

For the system (1),

$$K = 1/2(E - V)^2\{(2 - \alpha)(q_1^2 + q_2^2) + 1/(E - V)[(\partial V/\partial q_1)^2 + (\partial V/\partial q_2)^2]\}. \tag{17}$$

One can see from (17) that K is always positive implying local stability. But we know that the system is nonintegrable except for $\alpha = 0$.

5. Potential boundary

In the Riemannian curvature calculation we did not include the potential boundary given by $E = V$. Now let us consider the potential boundary given by

$$\frac{(1-\alpha)}{12}(q_1^4 + q_2^4) + \tfrac{1}{2}q_1^2 q_2^2 = E. \tag{18}$$

Extrinsic curvature of the curve (18) is given by

$$R = \frac{4E}{3}\frac{[(1-\alpha)^3 - 3]q_1^2 q_2^2 + (1-\alpha)(q_1^4 + q_2^4)}{[(1-\alpha)^2/9(q_1^6 + q_2^6) + (5 - 2\alpha)/3(q_1^2 + q_2^2)q_1^2 q_2^2]^{3/2}}. \tag{19}$$

When $\alpha = 0$, R is positive for all values of q_1 and q_2. R is negative in between the points of the boundary (q_1, q_2) and (q'_1, q'_2). Because of symmetry we consider only the first quadrant

$$q_1 = \frac{E^{1/4}}{[(1-\alpha)/12(1 + p^4) + 1/2p^2)^{1/4}},$$
$$q_2 = p q_1, \tag{20}$$
$$q'_1 = p q'_2,$$

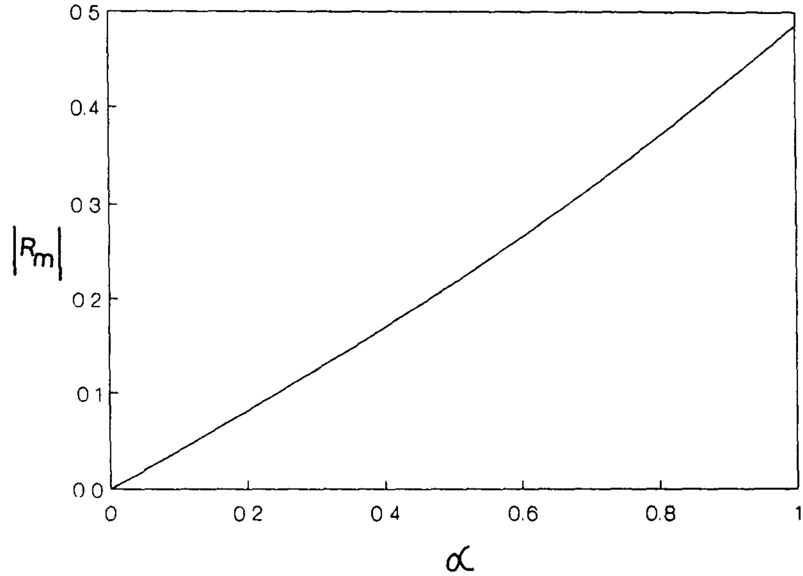

Figure 3. Plot of maximum curvature $|R_m|$ vs α.

$$q_2' = \frac{E^{1/4}}{[(1-\alpha)/12(1+p^4) + 1/2p^2]^{1/4}}, \tag{21}$$

where,

$$p^2 = \{3 - (1-\alpha)^2 - \{[(1-\alpha)^2 - 3]^2 - 4(1-\alpha)^2\}^{1/2}\}/2(1-\alpha). \tag{22}$$

When $q_1 = q_2$, R is the maximum and it is given by

$$R_m = \frac{-\alpha}{(4-\alpha)^{3/4}} \left(\frac{3}{2E}\right)^{1/4}. \tag{23}$$

In figure 3, $|R_m|$ versus α is plotted for energy $E = 1$.

On comparing figures 2 and 3 we can see that the chaos in the system is directly correlated to the negative curvature of the potential boundary.

6. Conclusion

In this paper we have presented a simple model in which there is connection between chaos and the curvature of the Riemannian manifold in which the evolution can be considered as a geodesic flow. Negative curvature implies chaos but positive curvature does not give rise to integrability. A chaotic quartic system is investigated which has strictly positive curvature. We calculate the LE and show that these are directly correlated with negative curvature of the potential boundary. As the negative boundary curvature increases chaos also increases in the system.

Savvidy [2] and Kawabe and Ohta [12] have observed a connection between the chaotic behaviour of certain Hamiltonian systems and of planar billiards with boundary as the potential boundary. In the case of billiards exponential instability arises due to the reflection at the negatively curved boundary. What happens at the reflection from the potential boundary is, however qualitatively different and it may appear that negative curvature of the boundary can have no direct link with the long term behaviour of a trajectory. But in understanding the chaotic behaviour what is important is not the behaviour of an individual trajectory but the behaviour of two nearby trajectories relative to one another. It is quite conceivable that the negative curvature of the boundary may have an influence over this even though we are not immediately able to suggest a mechanism by which the observed correlation between negative curvature and exponential divergence of trajectories may be explained. Further investigations are necessary to clarify the possible connection between the negative curvature of the potential boundary and chaotic behaviour and its relation to corresponding billiard problem.

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