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On the solution of an anisotropic nonquadratic potential

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Abstract. The Schrodinger–Green function is constructed for an anisotropic non-quadratic potential which has been studied in recent literature. The eigen energies and wavefunctions are readily obtained. Our analysis shows that the wavefunctions given in earlier literature are incorrect and the source of the error is pointed out. A semiclassical treatment of the problem is also presented in support of some of our observations.

Keywords. Schrödinger-Green function; nonquadratic potential.

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An interesting exactly solvable nonquadratic potential possessing neither spherical symmetry nor axial symmetry has been the subject of recent studies. In cartesian coordinates it has the form

$$V(x,y,z) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\beta}{y^2} + \frac{\gamma x}{y^2\sqrt{x^2+y^2}} + \alpha' z^2 + \frac{\beta'}{z^2}, \tag{1}$$

where α, β, γ, α', β' are all positive constants. The quantum mechanical eigenvalue problem for the above potential was solved by Cai and Inomata [1] by a path integral method. An algebraic approach to the same problem was given by Boschi-Filho and Vaidya [2], utilizing the dynamical symmetry group associated with the potential (1). In the present paper we construct the Schrödinger–Green's function for the above potential from which the eigen-energies and wavefunctions are readily deduced. Our construction proceeds as in the case of the Coulomb potential presented earlier [3]. Our results show that the wavefunctions given by both Cai and Inomata [1] and Boschi-Filho and Vaidya [2] are incorrect and we point out the source of error in these works. A second interesting observation following from our analysis is that the problem of the anisotropic non-quadratic potential (1) is really that of two uncoupled oscillators (in disguise!). This observation is further substantiated by us through a semi-classical treatment of the problem (with the usual WKB replacement for zero-point energy and Langer modification for radial problems).

The potential (1) is singular on the planes $y=0$ and $z=0$; therefore motion is restricted to one quarter of the three-dimensional space. We take this to be the region $y>0$, $z>0$. Since $V(x,y,z)=V_{12}(x,y)+V_3(z)$, the z-motion decouples from the xy-motion, and can be treated separately.

To obtain the quantum mechanical solution, we consider the Green function for the coupled xy-motion first. It obeys the equation

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + V_{12}(x,y) - \varepsilon\right] G_{12}(xy, x'y'; \varepsilon) = \delta(x-x')\delta(y-y'). \tag{2}$$

Introducing parabolic coordinates ξ and η defined by

$$x=\tfrac{1}{2}(\xi^2-\eta^2), \quad y=\xi\eta \tag{3}$$

such that $0\leqslant\eta<\infty$, $-\infty<\xi<\infty$, $\xi>0$ if $y>0$, equation (2) can be transformed to the form

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)-2\alpha+\frac{\lambda_1(\lambda_1+1)}{\xi^2}+\frac{\lambda_2(\lambda_2+1)}{\eta^2}-\varepsilon(\xi^2+\eta^2)\right]\tilde{G}= \delta(\xi-\xi')\delta(\eta-\eta') \tag{4}$$

where

$$\tilde{G}=\tilde{G}(\xi\eta,\xi'\eta';2\alpha)=G_{12}(xy,x'y';\varepsilon) \tag{5}$$

and

$$\lambda_1=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+8(\beta-\gamma)}, \quad \lambda_2=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+8(\beta+\gamma)}. \tag{6}$$

The structure of (4) enables us to identify $\tilde{G}$ as the Green function at energy 2α of a system of two identical oscillators of frequency $\omega=\sqrt{-2\varepsilon}$ with radial coordinates ξ and η, the oscillators having different fixed angular momenta λ_1 and λ_2.

Since the oscillators are noninteracting, we may write the following representation for $\tilde{G}$:

$$\tilde{G}=\sum_{n_1,n_2=0}^{\infty}\frac{u_{n_1\lambda_1}(\xi)u_{n_2\lambda_2}(\eta)u^*_{n_1\lambda_1}(\xi')u^*_{n_2\lambda_2}(\eta')}{E_{n_1}+E_{n_2}-2\alpha}. \tag{7}$$

In this, n_1, $n_2=0,1,2,\ldots$ are radial quantum numbers, and $u_{n_i\lambda_i}$ is the normalized reduced radial wavefunction of oscillator i, with the angular momentum analytically continued to the value λ_i. Such a continuation to nonintegral l is known to be valid for Re $l>-\frac{1}{2}$. E_{n_i} is the corresponding energy value given by

$$E_{n_i}=(2n_i+\lambda_i+\tfrac{3}{2})\sqrt{-2\varepsilon}. \tag{8}$$

It is a simple matter to verify that (7) satisfies (2). We note in passing that $\tilde{G}$ has also the integral representation

$$\tilde{G}=\lim_{\delta\to 0+} i\int_0^\infty \mathrm{d}T\exp[i(2\alpha+i\delta)T]K_{\lambda_1}(\xi,\xi';T)K_{\lambda_2}(\eta,\eta';T) \tag{9}$$

where $T=t-t'$ and the K_{λ_i} are the radial parts of the Feynman propagators of the two oscillators. For our present purposes, the eigenfunction expansion (7) is more convenient.

It is clear from (7) that $\tilde{G}$ has poles at α values given by $2\alpha=E_{n_1}+E_{n_2}$; hence G_{12} has poles at the energy values

$$\varepsilon=\varepsilon_{n_1n_2}=-\frac{2\alpha^2}{(2n_1+2n_2+\lambda_1+\lambda_2+3)^2}, \tag{10}$$

which are the allowed energies of the xy-motion. To this, we must add the allowed energies of the z-motion. Since the z-motion is that of an oscillator of frequency $\sqrt{2\alpha'}$ and angular momentum $\sqrt{2\beta'}$, the energy levels are given by

$$\varepsilon_{n_3}=\sqrt{2\alpha'}(2n_3+\lambda_3+\tfrac{3}{2}) \tag{11}$$

where $n_3=0,1,2,\ldots$ and λ_3 is defined by $\lambda_3(\lambda_3+1)=2\beta'$.

The wavefunction of the system is the product $\Psi = \psi(x,y)\phi(z)$. The function ψ is given by the residue of G_{12} at its poles:

$$|\psi|^2 = [(\varepsilon_{n_1 n_2} - \varepsilon)\tilde{G}(\xi\eta, \xi\eta; 2\alpha)]_{\varepsilon = \varepsilon_{n_1 n_2}}$$
$$= \frac{2\alpha}{(2n_1 + 2n_2 + \lambda_1 + \lambda_2 + 3)^2} |u_{n_1\lambda_1}(\xi) u_{n_2\lambda_2}(\eta)|^2. \tag{12}$$

Using the known form of the oscillator radial functions $u_{n\lambda}$, we deduce from (12) the explicit expression

$$\psi = \psi_{n_1 n_2} = \left(\frac{\bar{\omega}}{2n_1 + 2n_2 + \lambda_1 + \lambda_2 + 3} \right)^{1/2} \chi^{(\bar{\omega})}_{n_1\lambda_1}(\xi) \chi^{(\bar{\omega})}_{n_2\lambda_2}(\eta) \tag{13}$$

where

$$\chi^{(\bar{\omega})}_{n_i\lambda_i}(x) = \left[\frac{2\sqrt{\bar{\omega}} n_i!}{\Gamma(n_i + \lambda_i + \frac{3}{2})} \right]^{1/2} (\bar{\omega}x^2)^{1/2(\lambda_i + 1)} \exp(-\tfrac{1}{2}\bar{\omega}x^2)\, L^{\lambda_i + 1/2}_{n_i}(\bar{\omega}x^2) \tag{14a}$$

$$\bar{\omega} = \sqrt{-2\varepsilon_{n_1 n_2}} = 2\alpha[2n_1 + 2n_2 + \lambda_1 + \lambda_2 + 3]^{-1} \tag{14b}$$

and

$$\xi = [x + \sqrt{x^2 + y^2}]^{1/2} \geqslant 0, \quad \eta = [-x + \sqrt{x^2 + y^2}]^{1/2} \geqslant 0. \tag{15}$$

Using the properties of the associated Laguerre polynomials $L^{\lambda + 1/2}_n$, it can be verified that $\psi_{n_1 n_2}$ satisfies, as it should, the condition

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y |\psi_{n_1 n_2}|^2 = \int_0^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta (\xi^2 + \eta^2) |\psi_{n_1 n_2}|^2 = 1. \tag{16}$$

At this point we note that the wavefunction for xy-motion given by Cai and Inomata [1] does not satisfy the above normalization condition. Nor does it have the correct dimension. The same remarks apply also to the wavefunction of Boschi-Filho and Vaidya [2]. In these references the residue of G_{12} has not been taken in the energy variable of the xy-problem and the Jacobian in the normalization integral also seems to have been missed. The powers of u and v in (27) of ref. [2] are also incorrect.

As noted already, the z-motion is that of an oscillator of frequency $\omega' = \sqrt{2\alpha'}$ and fixed angular momentum λ_3 defined by $\lambda_3(\lambda_3 + 1) = 2\beta'$. The allowed energies are those given by (11), and the normalized wavefunctions are

$$\phi_{n_3}(z) = \chi^{(\omega')}_{n_3\lambda_3}(z) = \left[\frac{2\sqrt{\omega'} n_3!}{\Gamma(n_3 + \lambda_3 + \frac{3}{2})} \right]^{1/2}$$
$$\times (\omega' z^2)^{1/2(\lambda_3 + 1)} \exp(-\tfrac{1}{2}\omega' z^2)\, L^{\lambda_3 + 1/2}_{n_3}(\omega' z^2). \tag{17}$$

The complete energy Green function, including the z-part, can now be written down in the form

$$G(xyz, x'y'z'; E) = \sum_{n_1 n_2 n_3 = 0}^{\infty} \frac{\psi_{n_1 n_2}(x,y)\phi_{n_3}(z)\psi^*_{n_1 n_2}(x',y')\phi^*_{n_3}(z')}{E_{n_1 n_2 n_3} - E}, \tag{18}$$

where $E_{n_1n_2n_3}$ are the energies given by

$$E_{n_1n_2n_3} = \varepsilon_{n_1n_2} + \varepsilon_{n_3}. \tag{19}$$

We now present the semi-classical treatment of the problem starting from the classical Hamilton–Jacobi equation. For the potential V_{12} the Hamilton–Jacobi equation for xy-motion turns out to be separable in parabolic coordinates defined by eq. (3). In these coordinates V_{12} has the form

$$V_{12} = \frac{1}{\xi^2 + \eta^2}\left[-2\alpha + \frac{\beta+\gamma}{\eta^2} + \frac{\beta-\gamma}{\xi^2}\right] \tag{20}$$

while the kinetic energy (taking the mass to be unity) is

$$T_{12} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2). \tag{21}$$

From (20) and (21) we obtain the Hamiltonian

$$H_{12} = T_{12} + V_{12} = \frac{1}{\xi^2 + \eta^2}\left[\frac{1}{2}p_\xi^2 + \frac{1}{2}p_\eta^2 - 2\alpha + \frac{\beta-\gamma}{\xi^2} + \frac{\beta+\gamma}{\eta^2}\right]. \tag{22}$$

The Hamilton–Jacobi equation resulting from (22) is

$$\frac{1}{\xi^2 + \eta^2}\left[\frac{1}{2}\left(\frac{\partial W}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial \eta}\right)^2 - 2\alpha + \frac{\beta-\gamma}{\xi^2} + \frac{\beta+\gamma}{\eta^2}\right] = \varepsilon, \tag{23}$$

where W is Hamilton's characteristic function for the problem, and ε is the constant value of the Hamiltonian. If we now write $W(\xi,\eta) = W_1(\xi) + W_2(\eta)$, the above separates into two equations,

$$\left(\frac{\mathrm{d}W_1}{\mathrm{d}\xi}\right)^2 - 4\alpha + \frac{\lambda_1(\lambda_1+1)}{\xi^2} - 2\varepsilon\xi^2 = 2\mu, \tag{24a}$$

$$\left(\frac{\mathrm{d}W_2}{\mathrm{d}\eta}\right)^2 + \frac{\lambda_2(\lambda_2+1)}{\eta^2} - 2\varepsilon\eta^2 = -2\mu, \tag{24b}$$

where μ is a separation constant, and λ_1, λ_2 are given by

$$\lambda_1(\lambda_1+1) = 2(\beta-\gamma), \quad \lambda_2(\lambda_2+1) = 2(\beta+\gamma). \tag{25}$$

In order to determine the energy of the periodic motion of the system, we define in the usual manner two action variables

$$J_\xi = \oint p_\xi \mathrm{d}\xi, \quad J_\eta = \oint p_\eta \mathrm{d}\eta \tag{26}$$

corresponding to the two degrees of freedom. For periodic motion the energy can be expressed as a function of J_ξ and J_η. Since $p_\xi = \mathrm{d}W_1/\mathrm{d}\xi$, we have from (24a)

$$J_\xi = \oint \frac{\mathrm{d}\xi}{\xi}[2(\mu+2\alpha)\xi^2 - \lambda_1(\lambda_1+1) + 2\varepsilon\xi^4]^{1/2}. \tag{27}$$

With the substitution $t = \xi^2$ the integral in (27) becomes elementary, and we get

$$J_\xi = -\pi[\lambda_1(\lambda_1+1)]^{1/2} + \pi(\mu+2\alpha)(-2\varepsilon)^{-1/2}. \tag{28}$$

Using (24b) and (28), the other action variable is readily seen to be

$$J_\eta = -\pi[\lambda_2(\lambda_2+1)]^{1/2} - \pi\mu(-2\varepsilon)^{-1/2}. \tag{29}$$

Elimination of the separation constant μ between (28) and (29) leads to the following expression for the energy:

$$\varepsilon = -2\pi^2\alpha^2[J_\xi + J_\eta + \pi\sqrt{\lambda_1(\lambda_1+1)} + \pi\sqrt{\lambda_2(\lambda_2+1)}]^{-2}. \tag{30}$$

We now obtain from the above the semiclassical energy levels by imposing the (modified) Sommerfeld–Wilson quantization condition (with $\hbar = 1$)

$$J_\xi = 2\pi(n_1 + \tfrac{1}{2}), \quad J_\eta = 2\pi(n_2 + \tfrac{1}{2}), \tag{31}$$

n_1, n_2 being nonnegative integers, together with the Langer replacement

$$\lambda_i(\lambda_i+1) \to (\lambda_i + \tfrac{1}{2})^2, \quad i = 1, 2. \tag{32}$$

The result is

$$\varepsilon = \varepsilon_{n_1 n_2} = -\frac{2\alpha^2}{(2n_1 + 2n_2 + \lambda_1 + \lambda_2 + 3)^2} \tag{33}$$

which coincides with the exact quantum mechanical result (eq. (10)). For pedagogical reasons we may note that the non-central potential (1) is more easily solvable by Hamilton–Jacobi theory than by other methods.

In summary, the problem of the anisotropic nonquadratic potential (1) is just that of uncoupled harmonic oscillators in disguise.

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