Pramana – J. Phys., Vol. 39, No. 5, November 1992, pp. 501–508. © Printed in India.

Phase-function method for complex potentials

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MS received 15 July 1991; revised 16 April 1992

Abstract. We consider the scattering problem for absorptive interactions within the framework of phase-function method. A Green's function approach is used to derive the phase equation. As a case study we apply the algorithm presented on a shallow $\alpha-\alpha$ potential, the real and imaginary parts of which have been deduced from experimental data. The real and imaginary parts of the S-wave phase shift are found to vary smoothly with energy while those for D and G waves show some fluctuations in the low-energy region. It is shown that studies in spatial behaviour of the phase function provide a plausible explanation for the dynamical origin of these fluctuations.

Keywords. Scattering theory; phase-function method; complex potential; phase shifts.

PACS Nos 24·10; 25·10

1. Introduction

The computation of scattering phase shifts $\delta_l(k)$ as a function of centre of mass energy $E_{cm}(=k^2>0)$ constitutes one of the basic problems in scattering theory. Traditionally, this is done by integrating the lth partial wave radial Schrödinger equation from origin to the asymptotic region where the potential is negligible and then comparing the phase of the radial wavefunction with that of a comparison circular function. The phase-function method (PFM) (Calogero 1967) represents an alternative to the conventional Schrödinger-equation approach. The mathematical foundation of the method is a fact well known in the theory of differential equations that certain linear homogeneous equations of second order can be reduced to first order nonlinear equations of the Riccati type. In the PFM, the radial Schrödinger equation is reduced to a Riccati equation. The function which satisfies the Riccati equation (a phase function $\delta_l(k,r)$) has at each point the meaning of the phase shift of the wavefunction for scattering by the potential $V(r)$ cut-off at that point. Thus here the problem reduces to a direct determination of the desired scattering phase shift. At the same time it turns out that the knowledge of $\delta_l(k,r)$ when supplemented by an amplitude function $\alpha_l(k,r)$ is sufficient for the complete determination of the wavefunction. The function $\alpha_l(k,r)$ coincides with the modulus of the Jost function (Newton 1960) for the potential amputated of all parts extending beyond r.

The PFM has proven to be useful to deal with scattering problems in which potential functions are taken real and the corresponding Hamiltonian hermitian. In order to describe annihilation and absorption in certain processes, it is often convenient to work with models where the potential is no longer real. In fact, in any process with more than one open channel, the effect of one or more of the additional

channels can be taken into account formally by the introduction of a complex potential. In this paper, we shall consider the scattering on complex potential and examine the efficiency and usefulness of the PFM as applied in this case. It is interesting to note that the elementary theory of scattering remains valid even if the potential is complex. The only difference is that the phase shifts $\delta_l(k)$ are not real but have a positive imaginary part related to an absorption cross section according to

$$\sigma_a = \pi k^{-2} \sum_{l=0}^{\alpha} (2l+1)(1-\eta_l^2), \tag{1}$$

where

$$\eta_l = |S_l| = \exp(-2\,\mathrm{Im}\,\delta_l(k)). \tag{2}$$

As noted by Calogero (1967) in the phase method also no departure is required from the development of a real potential. Keeping this in view, we develop in § 2 the phase method for scattering on a complex absorptive interaction. We devote § 3 to demonstrate the usefulness of the constructed algorithms by dealing with a complex angular momentum dependent α–α potential (Darriulat *et al* 1965). In the past, one of us noted that (Talukdar *et al* 1983) the real part alone of this potential can well account for the elastic scattering phase shifts and the PFM help acquire a physical feeling for the relation between the potential and the off-shell extension function. Here one of our main objectives is to see how a weak imaginary part of the potential affects the elastic scattering phases.

2. PFM for complex potentials

Let us begin by considering the l-wave radial Schrödinger equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(u_l(k,r)) + [k^2 - l(l+1)/r^2]u_l(k,r) = V(r)u_l(k,r) \tag{3}$$

for real potential $V(r)$ for which the first and the second absolute moments (Newton 1982) exist. We have written (3) in units in which $\hbar^2/2m$ is unity. The regular free particle Green's function is given by

$$\left.\begin{aligned} G_l(r,r') &= -k^{-1}\{\hat{j}_l(kr)\hat{\eta}_l(kr') - \hat{j}_l(kr')\hat{\eta}_l(kr)\} && \text{for } r' < r \\ &= 0 && \text{for } r' > r \end{aligned}\right\}. \tag{4}$$

Here $\hat{j}_l(kr)$ and $\hat{\eta}_l(kr)$ stand for the Riccati–Bessel and Riccati–Neumann functions. In terms of the Green's function in (4) the differential equation can be converted to an integral equation

$$u_l(k,r) = \hat{j}_l(kr) - k^{-1}\int_0^r \mathrm{d}r'[\hat{j}_l(kr)\hat{\eta}_l(kr') - \hat{j}_l(kr')\hat{\eta}_l(kr)]\; V(r')u_l(k,r'). \tag{5}$$

In the PFM the phase and amplitude functions are introduced through the ansatz

$$\alpha_l(k,r)\cos\delta_l(k,r) = 1 - k^{-1}\int_0^r \mathrm{d}r'\,\hat{\eta}_l(kr')\; V(r')u_l(k,r') \tag{6a}$$

and

$$\alpha_l(k,r)\sin\delta_l(k,r) = -k^{-1}\int_0^r \mathrm{d}r'\hat{j}_l(kr')\,V(r')u_l(k,r') \tag{6b}$$

with $\delta_l(k,0)=0$ and $\delta_l(k,r_\infty)=\delta_l(k)$, the phase shift and $\alpha_l(k,0)=1$ and $\alpha_l(k,r_\infty)=\alpha_l(k)$. Here r_∞ is the radial distance at and after which the effect of the potential is negligible. The vanishing of the phase function $\delta_l(k,r)$ at $r=0$ implies that a completely amputated potential does not produce any phase shift. Also, we note that the amplitude function $\alpha_l(k,r)$ is a measure of focussing or defocussing of the projectile by the target potential $V(r)$ (Fano *et al* 1976). Thus it is physically expected that $\alpha_l(k,0)=1$. Using (6) in (5) we have

$$u_l(k,r) = \alpha_l(k,r)[\hat{j}_l(kr)\cos\delta_l(k,r) - \hat{\eta}_l(kr)\sin\delta_l(k,r)]. \tag{7}$$

From (6) and (7) one can write the phase equation as

$$\frac{\mathrm{d}}{\mathrm{d}r}(\delta_l(k,r)) = -k^{-1}\,V(r)[\hat{j}_l(kr)\cos\delta_l(k,r) - \hat{\eta}_l(kr)\sin\delta_l(k,r)]^2. \tag{8}$$

Since the formal scattering theoretic equations remain valid even when $V(r)$ is complex, we can write

$$\frac{\mathrm{d}}{\mathrm{d}r}(\mathrm{Re}\,\delta_l(k,r) + i\,\mathrm{Im}\,\delta_l(k,r)) = -k^{-1}(V_R(r) + iV_I(r))$$
$$[\hat{j}_l(kr)\cos(\mathrm{Re}\,\delta_l(k,r) + i\,\mathrm{Im}\,\delta_l(k,r)) - \hat{\eta}_l(kr)\sin(\mathrm{Re}\,\delta_l(k,r) + i\,\mathrm{Im}\,\delta_l(k,r))]^2. \tag{9}$$

In writing (9) we have used a complex value of the phase function for scattering on complex potential

$$V(r) = V_R(r) + iV_I(r). \tag{10}$$

With the present computing facilities (9) can be solved as it stands with the initial condition $\mathrm{Re}\,\delta_l(k,0)=0$ and $\mathrm{Im}\,\delta_l(k,0)=0$. However, the real and imaginary parts of (9) can be separated to reproduce the standard results of Calogero (1967) obtained by using an interpolating S-matrix function written as

$$S_l(k,r) = \exp[2i(\mathrm{Re}\,\delta_l(k,r) + i\,\mathrm{Im}\,\delta_l(k,r))]. \tag{11}$$

In the next section we shall employ (9) to compute the scattering phase shifts produced by a complex $\alpha-\alpha$ potential and thus clarify certain conceptual aspects of the scattering theory for absorptive interaction.

3. A case study

In an attempt to fit the $\alpha-\alpha$ scattering phase shifts for laboratory energies between 50 and 120 MeV Darriulat *et al* (1965) introduced an angular momentum-dependent complex Saxon-Woods potential of the form

$$V_{\alpha\alpha}(r) = u_1\{1+\exp[(r-r_1)/a_1]\}^{-1} - u_2\{1+\exp[(r-r_2)/a_2]\}^{-1}$$
$$- iW\{1+\exp[(r-r_3)/a_3]\}^{-1}. \tag{12}$$

The first term represents the repulsive core and the second term is a larger range attractive potential. The third term takes care of the inelastic processes, if any. The parameters of the potential (12) are given in table 1. Looking at this table we see that the imaginary part of the potential is only a few per cent of its real part. Thus the potential in (12) provides a natural basis to gain some physical weight for the problem of adding a small imaginary part to an otherwise satisfactory real potential (Talukdar *et al* 1983) even below $E_{lab} = 50$ MeV.

The results for $\delta_l(k)$ were obtained from (9) by using the algorithms of the Runge-Kutta method with an appropriate stability check. In particular, it was noted that the repulsive part of the potential has an extremely small diffuseness and is therefore a rapidly varying function. It was made sure that no oscillations in the values of $\delta_l(k)$ occur due to inadequate numerical analysis of the integration technique. Despite that we observed some oscillations in the values of $\mathrm{Im}\,\delta_l(k)$. We venture to suggest these as arising from the absorptive part of the potential.

In figure 1 we portray $\mathrm{Re}\,\delta_l(k)$ and $\mathrm{Im}\,\delta_l(k)$ as a function of centre of mass energy E_{cm} for the partial waves $l = 0$, 2 and 4. The imaginary part of the phase shifts for the S-wave vary smoothly with E_{cm}. This is true for D and G waves only after $E_{cm} = 22$ MeV. In the low-energy region $\mathrm{Im}\,\delta_2(k)$ and $\mathrm{Im}\,\delta_4(k)$ exhibit some rapid

Table 1. Parameters for the potential in (12).

l	u_1(MeV)	u_2(MeV)	W(MeV)	r_1(fm)	r_2(fm)	$r_2 = r_3$(fm)	$a_2 = a_3$(fm)
0	150	9·2	5	1·65	0·1	3·72	0·4
2	150	16	5	1·63	0·05	3·55	0·3
4	220	71	5	1·20	0·05	2·48	0·46

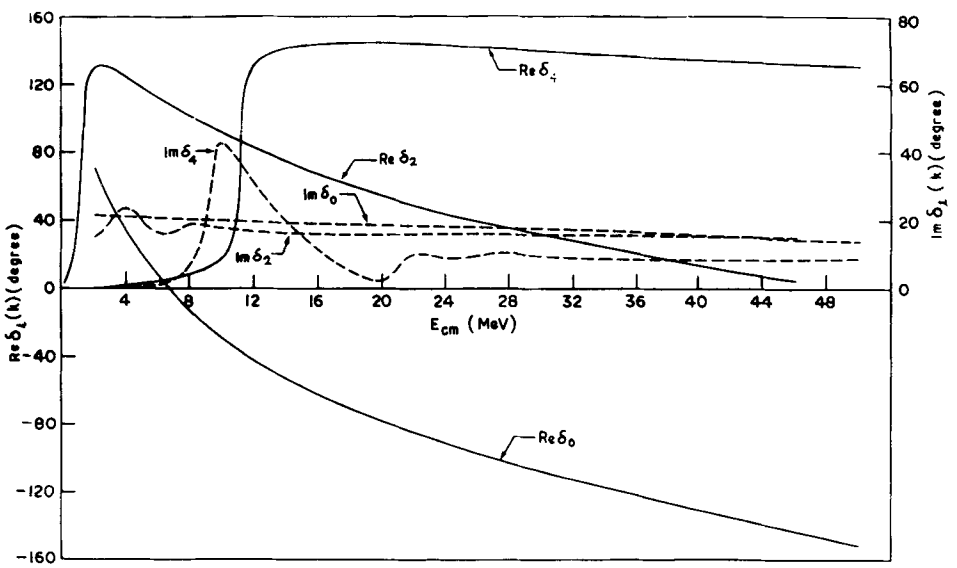

Figure 1. The real and imaginary parts of the phase shift $\delta_l(k)$ as a function of centre of mass energy E_{cm}. The ordinate for $\mathrm{Re}\,\delta_l(k)$ is taken on the left while that for $\mathrm{Im}\,\delta_l(k)$ on the right.

fluctuations. One would be tempted to acknowledge them as symptomatic of sharp resonances. But we know that no such resonances do exist in the low-energy region for the $\alpha-\alpha$ scattering. Thus great care must be taken while dealing with scattering on a complex potential. During the course of our computation we found that remarkable smooth and systematic behaviour of the real part of the phase shift was destroyed in the energy regions where $\mathrm{Im}\,\delta_l(k)$ fluctuated rapidly. However, the real part of the phase shifts above and below these narrow ranges joined smoothly leaving no trace on the cross-section.

An added advantage of the PFM is that one can study the behaviour of $\delta_l(k,r)$ as a function of r in producing the scattering phase shift. This can be exploited to examine how the peaks in $\mathrm{Im}\,\delta_l(k,r)$ modulate the behaviour of $\mathrm{Re}\,\delta_l(k,r)$ for apparent discontinuities in the values of $\mathrm{Re}\,\delta_l(k,r)$ at certain energies. Let us concentrate our

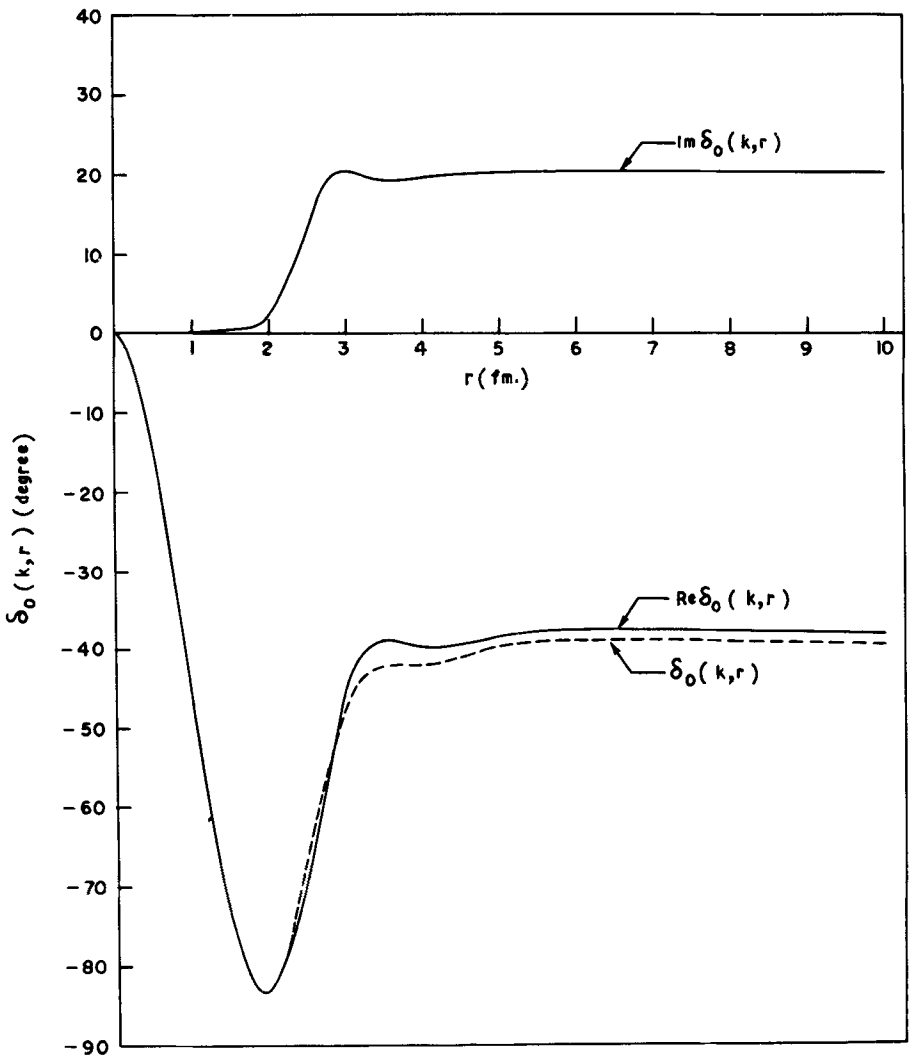

Figure 2. The real and imaginary parts of the phase function $\delta_0(k,r)$ as a function of r at $E_{cm} = 12$ MeV. The dashed curve represents the variation of $\delta_0(k,r)$ when $W = 0$.

attention on the partial waves $l = 0$ and $l = 4$ at $E_{cm} = 12$ MeV. Near this energy both $\mathrm{Re}\,\delta_0(k)$ and $\mathrm{Im}\,\delta_0(k)$ exhibit well established characteristics while $\mathrm{Re}\,\delta_4(k)$ and $\mathrm{Im}\,\delta_4(k)$ do not. Thus by making a comparison between the spatial behaviour of the appropriate phase functions we can hope to gain some physical weight for the anomalous behaviour of $\delta_4(k)$.

In figures 2 and 3 we portray the variation of $\delta_0(k,r)$ and $\delta_4(k,r)$ as a function of r. Looking closely into these curves we see that $\mathrm{Im}\,\delta_0(k,r)$ and $\mathrm{Im}\,\delta_4(k,r)$ have positive values for all r. This is quite expected for a potential with $\mathrm{Im}\,V(r) < 0$ (Thylwe and Fröman 1983). The phase function $\mathrm{Im}\,\delta_0(k,r)$ is a monotonically increasing function of r up to $r = 3$ fm. Then it saturates smoothly to produce the phase shift $\mathrm{Im}\,\delta_0(k)$. The values of $\mathrm{Re}\,\delta_0(k,r)$ and $\delta_0(k,r)$ [phase function with $W = 0$] are not discernible

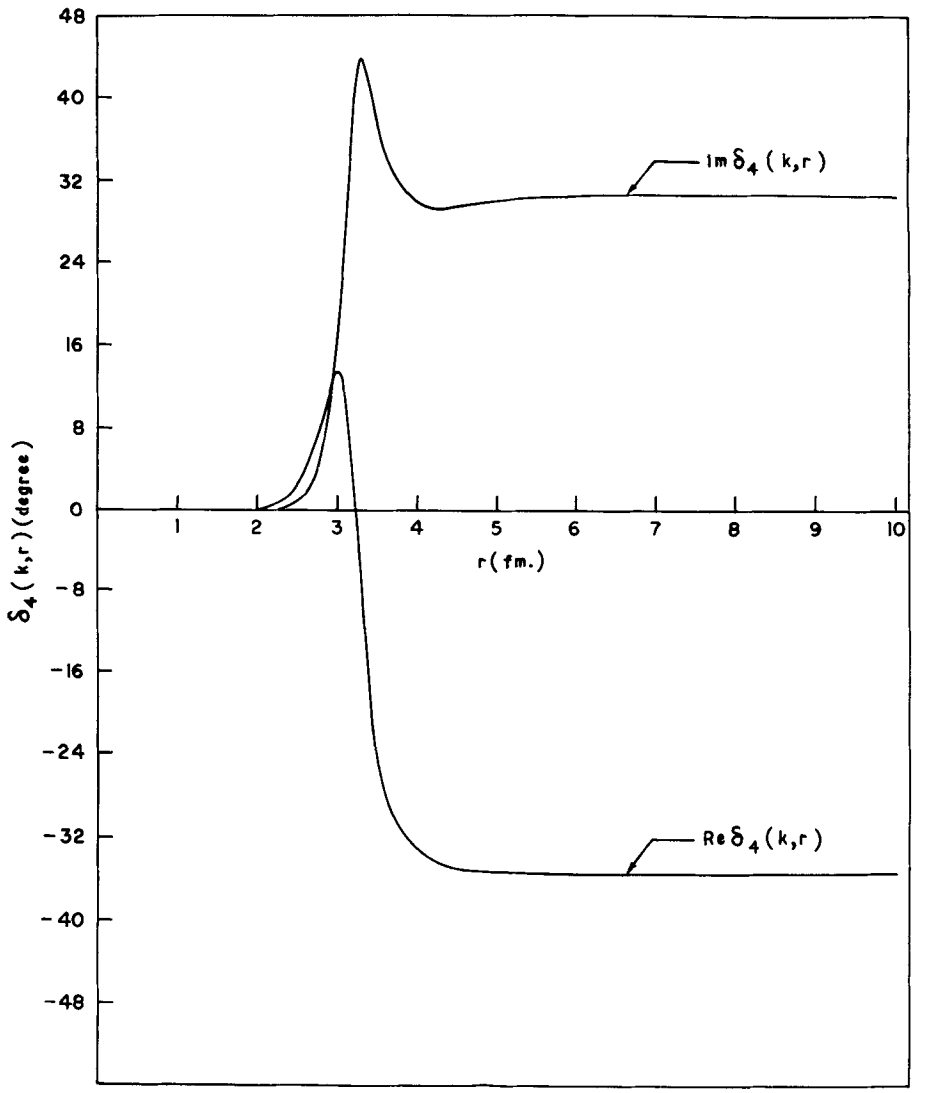

Figure 3. The real and imaginary parts of the phase function $\delta_4(k,r)$ as a function of r at $E_{cm} = 12$ MeV.

up to $r = 2{\cdot}8$ fm. Also we note that $|\mathrm{Re}\,\delta_0(k) - \delta_0(k)|$ is only a few per cent of the value of $|\mathrm{Re}\,\delta_0(k)|$ or $|\delta_0(k)|$. Thus the S-wave scattering phase shift at $E_{cm} = 12$ MeV does not appear to be significantly affected by the imaginary part of the potential. In contrast to this the behaviour of $\delta_4(k,r)$ shown in figure 3 is somewhat unwanted. For example, $\mathrm{Im}\,\delta_4(k,r)$ has a sharp peak near $r = 3$ fm. where $\mathrm{Re}\,\delta_4(k,r)$ takes up a large negative value giving ultimately $\mathrm{Re}\,\delta_4(k) = -36°$. But as with the S-wave case the correct value of $\mathrm{Re}\,\delta_4(k)$ should lie close to 130° as obtained from figure 4 which displays $\delta_4(k,r)$ [with $W = 0$] as a function of r. Also note that our interpolated value of $\mathrm{Re}\,\delta_4(k)$ in figure 1 is 132°. Thus the phase jump induced by the absorptive potential $\sim \pi$ radian. This type of rapid and discontinuous phase change can occur only in the vicinity of an isolated compound resonance (Mulligan *et al* 1976; Klaus 1991). The analysis presented here clearly demonstrates how an unphysical resonance can be induced by an apparently satisfactory complex potential.

The alpha-particle is tightly bound such that at least 20 MeV of energy is required in its centre of mass to loosen this "robust" object. Thus, up to about this energy of

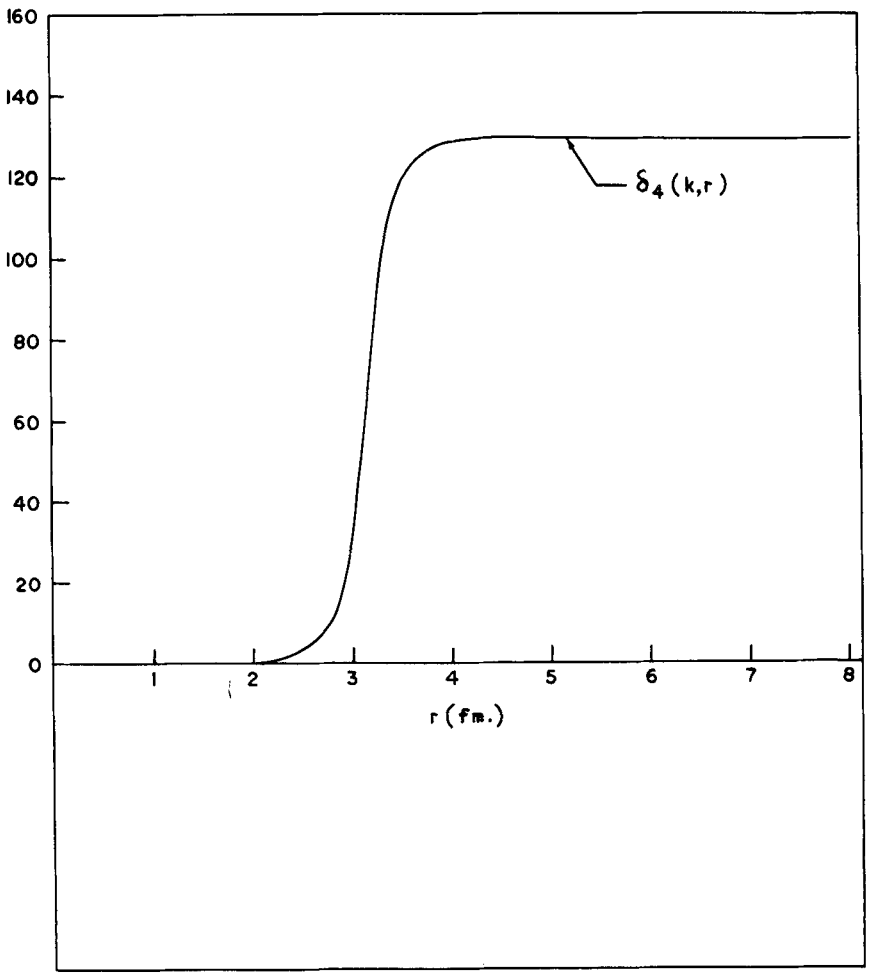

Figure 4. The phase function $\delta_4(k,r)$ as a function of r at $E_{cm} = 12$ MeV when $W = 0$.

the $\alpha - \alpha$ system, the scattering is elastic and the introduction of an imaginary potential may appear aesthetically unpleasing. But the present case study refers to the following.

The alpha-particle is believed to be an important substructure of more complex nuclei and the cluster model of nuclei and hypernuclei is of considerable interest. Such models involve two or more alphas and one needs to use a potential that is in accord with fundamental studies and scattering results. In this case external $\alpha - \alpha$ excitation will be feasible even at low energies and our model calculation refers to such a physical situation.

Acknowledgement

This work was supported in part by the Department of Atomic Energy, and CSIR, Government of India.

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