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Quantum cosmology in $R^1 \times S^1 \times S^n$ space time

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Abstract. Classical and quantum cosmological aspects for $(n+2)$ dimensional anisotropic spherically symmetric space-time with topology of $(n+1)$ space $S^1 \times S^n$ have been studied. The Lorentzian field equations are reduced to an autonomous system by a change of field variables and are discussed near the critical points. The path integral expression for propagation amplitude is converted to a single ordinary integration over the lapse function by the usual technique and is evaluated in terms of Bessel functions.

Keywords. Higher dimension; spherically symmetric; quantum propagator; saddle point method.

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1. Introduction

Cosmological phenomenon in higher dimension is getting much interest in recent years as superstring theory (Witten 1984) and Yang-Mills supergravity in their field theory limit require more than four-dimensional space-time. Also higher dimensional theories are an attractive way to unify gravitation and gauge interactions. Since at present our world is a four-dimensional space-time, the extra dimensions should be imperceptible. A possible mechanism to achieve this condition is the dynamical compactification of the extra dimensions (Chodos and Detweiler 1980). Incidentally, the entropy production and the inflation could be explained as a result of the contraction of extra dimensions (Alvarez and Gavela 1983; Abbott *et al* 1984). Further the experimental detection of time variation of fundamental constants supports the above assertion (Marciano 1984).

Quantum cosmology has been an important and useful field since the early stages of evolution of the universe. In particular, Euclidean quantum cosmology got much interest with developments in wormhole physics and the two thrilling proposals of wave functions of the universe—one due to Hartle-Hawking (1983) and the other due to Vilenkin (1984) and Linde (1984). So it is interesting to study quantum cosmology in higher dimension. In this paper we shall study cosmological phenomenon from both classical and quantum aspects in a $(n+2)$ dimensional homogeneous, anisotropic and spherically symmetric space-time with a cosmological constant.

The model is described in § 2 and Wheeler-DeWitt (W-D) equation is constructed and a particular solution evaluated. In § 3, the classical Lorentzian field equations are transformed to first order autonomous system and critical points are analysed. The path integral expression for propagation amplitude is simplified to an ordinary integral over lapse function by the usual technique in § 4. This lapse integral is then evaluated by steepest descent method using Bessel functions. Summary and conclusion are given in § 5.

2. The basic and W-D equations

The Euclidean metric for the $(n+2)$ dimensional homogeneous, anisotropic and spherically symmetric space-time with topology of $(n+1)$ space $S^1 \times S^n$ is

$$dS^2 = \rho^2(N^2 d\tau^2 + a^2(\tau)dr^2 + b^2(\tau)dQ_n^2), \tag{1}$$

where

$$dQ_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_1 \cdots + \sin^2\theta_{n-1} d\theta_n^2$$

is the metric on unit n-sphere. Now the Euclidean action for the gravitational field with a cosmological constant in the background of the above metric is

$$I = \frac{1}{2}\int d\tau N\left[-(n-1)ab^{n-2} - \frac{(n-1)}{N^2}ab^{n-2}b'^2 - 2b^{n-1}\frac{a'b'}{N^2} + \lambda ab^n\right].$$

The gauge transformation $\bar{N} = aN$, and the change of variable $C = a^2 b^{n-1}$, simplify the above action to

$$I = \frac{1}{2}\int d\tau \bar{N}\left[-(n-1)b^{n-2} - \frac{b'c'}{\bar{N}^2} + \lambda b^n\right]. \tag{2}$$

Now, variation of this action with respect to the field variables b, c and $\bar{N}$ gives the field equations

$$b'' = 0, \quad c'' = \bar{N}^2[(n-1)(n-2)b^{n-3} - 2n\lambda b^{n-1}], \tag{3}$$

and the constraint

$$-(n-1)b^{n-2} + \frac{b'c'}{\bar{N}^2} + \lambda b^n = 0. \tag{4}$$

The W-D equation for the model is

$$\left[\frac{\partial^2}{\partial b \partial c} + \lambda b^n - (n-1)b^{n-2}\right]\psi(b,c) = 0. \tag{5}$$

Using separation of variables, the solution can be written as

$$\psi(b,c) = A\exp\left[\mu\left(\frac{\lambda b^{n+1}}{n+1} - b^{n-1}\right) - c/\mu\right]$$

with μ as a separation constant and A an arbitrary constant.

3. Lorentzian field equations

The Lorentzian $(n+2)$ metric for the anisotropic, homogeneous and spherically symmetric space-time is

$$dS^2 = \rho^2(-N^2 dt^2 + a^2(t)dr^2 + b^2(t)d\Omega_n^2). \tag{6}$$

So the Lorentzian action of the system is

$$A = \frac{1}{2}\int N\,dt\left[(n-1)ab^{n-2} - \frac{(n-1)}{N^2}ab^{n-2}b'^2 - \frac{b^{n-1}a'b'}{N^2} - \lambda ab^n\right]. \tag{7}$$

Hence the Lorentzian field equations are

$$\frac{(n-1)}{b^2} + 2\frac{a'b'}{ab} + (n-1)\frac{b'^2}{b^2} - \lambda = 0, \tag{8}$$

$$2\frac{b''}{b} + (n-1)\frac{b'^2}{b^2} + \frac{(n-1)}{b^2} - \lambda = 0, \tag{9}$$

and

$$2\frac{a''}{a} + 2(n-1)\frac{b''}{b} + 2(n-1)\frac{a'}{a}\frac{b'}{b} + (n-1)(n-2)\frac{b'^2}{b^2} + \frac{(n-1)(n-2)}{b^2} - n\lambda = 0. \tag{10}$$

The change of variables

$$u = \alpha' = \frac{a'}{a}, \quad v = \beta' = \frac{b'}{b} \tag{11}$$

reduces the field equations to a first order system as

$$u' = -u^2 + \frac{\lambda}{2} + \frac{(n-1)}{2}v^2 - (n-1)uv + \frac{(n-1)}{2}\exp(-2\beta), \tag{12}$$

$$v' = \frac{\lambda}{2} - \frac{(n+1)}{2}v^2 - \frac{(n-1)}{2}\exp(-2\beta), \tag{13}$$

and the constraint

$$(n-1)(v^2 + \exp(-2\beta)) + 2uv - \lambda = 0. \tag{14}$$

Thus the system of first order equations (12) and (13) and the relation $\beta' = v$ together form a first order autonomous system in the variables (u, v, β). It has two critical points

$$E_1 : u = (\lambda)^{1/2}, \quad v = 0, \quad \beta = -\tfrac{1}{2}\ln(\lambda/n-1),$$

$$E_2 : u = -(\lambda)^{1/2}, \quad v = 0, \quad \beta = -\tfrac{1}{2}\ln(\lambda/n-1). \tag{15}$$

The roots of the characteristic equation show that the critical point E_1 is stable and is called inward nil potent vortex, while E_2 the unstable is called outward nil potent vortex. It should be noted that both the critical points lie on the constraint surface (14).

The solutions to the linearized system around the stable vortex are of the form

$$u=(\lambda)^{1/2}+\exp(-2(\lambda)^{1/2}t)-\left(\frac{nc_1}{3}\exp((\lambda)^{1/2}t)+(n-2)c_2\exp(-(\lambda)^{1/2}t)\right),$$

$$v=c_1\exp((\lambda)^{1/2}t)+c_2\exp(-(\lambda)^{1/2}t), \tag{16}$$

where c_1, c_2 are arbitrary constants.

In this degenerate case, E_1 and E_2 have purely imaginary pairs of complex eigen values. So there may be a local manifold containing periodic solutions of the non-linear equations. It is termed as a central manifold and bifurcates the neighbourhood of the fixed point E_1 into two branch surfaces. Also solution to the linearized system around the stable critical point which is an attractor, shows an exponential inflation for the system.

4. Quantum propagator: Path integral and its evaluation

As usual, the path integral expression for the quantum propagation amplitude between fixed initial and final values of the field variables is (Halliwell and Louko 1990)

$$G(b'',c''/b',c')=\int \mathrm{d}\bar{N}\int \mathscr{D}\ell\,\mathscr{D}c\exp(-I[b(\tau),c(\tau),\bar{N}]). \tag{17}$$

Here the action I has the expression in (2) and the functional integrals are taken over the sum of histories ($b(\tau)$, $c(\tau)$, $\bar{N}$) satisfying the boundary condition

$$b(0)=b',\quad c(0)=c',\quad b(1)=b'',\quad c(1)=c''. \tag{18}$$

The saddle points of the functional integrals are the solutions of the classical field equations (3) with the above boundary conditions (Chakraborty 1991) and are given by

$$\bar{b}=u\tau+b',$$

$$\bar{c}=\frac{\bar{N}^2}{u^2}(u\tau+b')^{n-1}-\frac{2\lambda\bar{N}^2}{u^2(n+1)}(u\tau+b')^{n+1}+c_1\tau+c_2, \tag{19}$$

where

$$u=b''-b',\quad c_2=c'-\frac{\bar{N}^2}{u^2}(b')^{n-1}+\frac{2\lambda\bar{N}^2}{u^2(n+1)}(b')^{n+1}$$

and

$$c_1=(c''-c')-\frac{\bar{N}^2}{u^2}((b'')^{n-1}-(b')^{n-1})+\frac{2\lambda\bar{N}^2}{u^2(n+1)}((b'')^{n+1}-(b')^{n+1}).$$

In order to evaluate the functional integral we introduce the following change of variables

$$b(\tau)=\bar{b}(\tau)+x(\tau),\quad c(\tau)=\bar{c}(\tau)+y(\tau). \tag{20}$$

The action in terms of these new variables may be written as

$$I[b(\tau),c(\tau),\bar{N}]=I_0(b'',c'',\bar{N}/b',c',0)+I_2[x(\tau),y(\tau),\bar{N}]. \tag{21}$$

The zeroth order action I_0 is the action at the saddle points and has the expression

$$I_0 = A\bar{N} + B/\bar{N}, \tag{22}$$

with

$$A = -\frac{(b'')^{n-1} - (b')^{n-1}}{2u} + \frac{\lambda}{u(n+1)}((b'')^{n+1} - (b')^{n+1}),$$

$$B = -(b'' - b')(c'' - c')/2.$$

The expression for the second order action I_2 is

$$I_2 = \frac{1}{2}\int_0^1 \bar{N}\mathrm{d}\tau\left[-(n-1)\bar{b}^{n-4}x^{2n-2}C_2 - \frac{x'y'}{\bar{N}^2} + nc_2\bar{b}^{n-2}x^2\right]. \tag{23}$$

Hence the propagation amplitude in terms of the new variables is

$$G(b'', c''/b', c') = \int \mathrm{d}\bar{N}\exp(-I_0)\int \mathscr{D}x\mathscr{D}y\exp(-I_2). \tag{24}$$

One should note that I_2 is quadratic in x and derivatives in x and y and so the functional integral over x and y in (24) are gaussian in nature. Therefore, using the standard results (with known convergent contours) the above expression for propagation amplitude reduces to a single ordinary integration over lapse function as (Chakraborty 1991)

$$G(b''c''/b'c') = \int \frac{\mathrm{d}\bar{N}}{\bar{N}}\cdot\exp[-I_0(b'', c'', \bar{N}/b', c', 0)]. \tag{25}$$

Here an overall multiplicative constant factor is neglected. The expression (25) is a solution of W-D equation if the contour is extended over infinite range or a closed loop, but for semi-infinite contour the propagation amplitude is a Green's function of the W-D operator. It is worthy to mention at this point that the same range of the Gauge-fixed lapse in the path integral need not be invariant under lapse rescalings in the action.

Now, the saddle point method will be employed to evaluate the above lapse integral. The saddle points are the values of $\bar{N}$ for which $\partial I_0/\partial\bar{N} = 0$ and are given by

$$\bar{N}^2 = B/A. \tag{26}$$

One should note that exact solution of the Einstein field equations with boundary conditions (18) is the saddle points (19) with $\bar{N}$ from (26). These solutions are real Euclidean if $A\cdot B > 0$ and it will be real Lorentzian if $A\cdot B < 0$.

The evaluation of the integral in (25) by saddle point method for different convergent contours has been studied extensively by Halliwell and Louko (1990). So we shall only describe the results, their asymptotic forms and corresponding contours in table 1. The asymptotic forms are valid for large $|AB|$. In this case the W-D operator is

$$H = \frac{1}{2}\left[\frac{\partial^2}{\partial b\partial c} + \lambda b^n - (n-1)b^{n-2}\right]. \tag{27}$$

Table 1. Results of evaluation of the lapse integral (25) by saddle point method for different convergent contours. Origin is the essential singularity for these contours.

Region	Contour	Quantum propagator $G(b'', c''/b', c')$	Asymptotic form	Nature of the solution
Euclidean ($A\cdot B>0$)	A closed contour around origin and through the saddle points $\bar{N} = \pm(B/A)^{1/2}$	$2\pi i I_0(\sqrt{AB})$	$\exp(\sqrt{AB})$	Solution of the W-D Equation
	The semi-infinite real axis starting from origin (+ve real axis if $A, B<0$ and −ve real axis if $A, B>0$)	$2K_0(\sqrt{AB})$	$\exp(-\sqrt{AB})$	Green function of the W-D operator
Lorentzian ($A\cdot B<0$)	A closed contour around origin and through the saddle points $\bar{N} = \pm i(-B/A)^{1/2}$	$2\pi i J_0(\sqrt{-AB})$	$\cos(\sqrt{-AB}-\pi/4)$	Solution of the W-D equation
	The semi-infinite imaginary axis starting from origin	$\pi i H_0^{(1)}(\sqrt{-AB})$	$\exp(i\sqrt{-AB})$	Green function of the W-D operator

So the above solutions will satisfy

$$H\cdot G(b'', c''/b', c') = 0 \text{ or } \delta(a''/b')\cdot\delta(c''/c'), \tag{28}$$

according as G is a solution or Green function of (27).

5. Summary and conclusion

The Euclidean-Einstein and the W-D equations are constructed in § 2. A particular solution of the W-D equation has been evaluated using separation of variables. In § 3 we deal with Lorentzian field equations. An autonomous system of first order equations is constructed from the second order Lorentzian field equations by a transformation of field variables. The critical points which are saddle in nature, are analysed in the linearized system. Quantum propagator has been studied in § 4. The path integral expression for propagation amplitude is reduced to a single ordinary integral over lapse function by the lapse method (Halliwell and Hartle 1990) and the concept of microsuperspace (Halliwell and Louko 1989) and one can obtain identical expression for propagation amplitude (except for constant multiplicative factor) as in anisotropic homogeneous spherically symmetric four dimensional space-time (Halliwell and Louko 1990). So table 1 gives only the results.

From the study of $(n+2)$-dimensional anisotropic spherically symmetric space-time one may conclude that the dimension of space-time does not depend sensitively on the results of both classical and quantum cosmological phenomena. The critical points and their nature are not affected by variation of n. Also the saddle points for the lapse integral in the expression for propagation amplitude may change due to n variation but nature is independent of n variation. Therefore, conclusion on cosmological phenomena depends on the topology of space but not on the dimension of the space.

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