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New forms of quantum statistics

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Abstract. We propose a new two-parameter deformation of the algebra of creation and destruction operators, which allows the construction of a new family of Hillbert spaces with positive definite inner product. This provides a continuous interpolation between two new forms of statistics named orthofermi and orthobose statistics. Positivity of the inner product over the two-parameter region is discussed.

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Recently, attention has been devoted to the interpolation between Bose and Fermi statistics using q-deformation of the Heisenberg algebra of the creation and destruction operators (Greenberg 1991; Fivel 1990; Mohapatra 1990; Chaturvedi *et al* 1991). In this note we propose a new two-parameter generalization of the algebra leading to two new forms of quantum statistics named orthofermi and orthobose statistics. A new family of Hilbert spaces with a positive-definite metric is constructed which interpolates analytically between the Fock spaces representing orthofermi and orthobose particles.

The generalized commutation relation which we propose is

$$c_{k\alpha}c^{\dagger}_{m\beta} - (q_1 - q_2)c^{\dagger}_{m\beta}c_{k\alpha} - q_2\delta_{\alpha\beta}\sum_{\gamma}c^{\dagger}_{m\gamma}c_{k\gamma} = \delta_{km}\delta_{\alpha\beta} \tag{1}$$

where c and $c^{\dagger}$ are destruction and creation operators and q_1 and q_2 are real parameters. In a physical problem, the latin indices $k, m\ldots$ and the greek indices $\alpha, \beta, \gamma\ldots$ may correspond to space and spin indices respectively and so we may call them accordingly. In general, for arbitrary q_1 and q_2 there is no commutation rule on cc or $c^{\dagger}c^{\dagger}$.

For $q_2 = 0$ and $q_1 = \mp 1$, the above reduces to Fermi and Bose statistics. The case $q_1 = q_2 = -1$ corresponds to the orthofermi statistics introduced in an earlier paper (Mishra and Rajasekaran 1991) and the case $q_1 = q_2 = +1$ can be called the orthobose statistics. These four cases of Fermi, Bose, orthofermi and orthobose statistics are the four corners of a parallelogram in $\{q_1, q_2\}$ space (see figure 1) such that on the two diagonals of the parallelogram, we have two families of algebras for which Hilbert-spaces with positive definite metric exist, one interpolating between Fermi

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and Bose statistics and the other interpolating between orthofermi and orthobose statistics. The former was studied in earlier literature (Greenberg 1991; Fivel 1990) and the latter is a new one.

We assume the existence of a vacuum state $|0\rangle$ annihilated by all the annihilators:

$$c_{k\alpha}|0\rangle = 0. \tag{2}$$

The Fock space is constructed in the obvious way. We consider the set of states which are linear combinations with complex coefficients of the monomials $c^{\dagger}_{k\alpha}c^{\dagger}_{m\beta}\ldots c^{\dagger}_{j\rho}|0\rangle$ and their duals $\langle 0|c_{p\gamma}\ldots c_{m\beta}$. Their inner product $\langle 0|C_{p\gamma}\ldots c^{\dagger}_{j\rho}|0\rangle$ or in fact the vacuum to vacuum matrix element of any polynomial in the c's and $c^{\dagger}$'s arbitrarily ordered, can be calculated using (1) and (2). No commutation rule on cc or $c^{\dagger}c^{\dagger}$ is required for this.

We must enquire into the restrictions imposed by the positivity of the inner product. We first do this for the two-particle sector. Consider the two-particle states with distinct space labels ($k \neq m$) and distinct spin labels ($\alpha \neq \beta$). (If these are not distinct, we get only weaker conditions). There are four states:

$$|1\rangle = c^{\dagger}_{k\alpha}c^{\dagger}_{m\beta}|0\rangle,\ |2\rangle = c^{\dagger}_{m\alpha}c^{\dagger}_{k\beta}|0\rangle,$$
$$|3\rangle = c^{\dagger}_{m\beta}c^{\dagger}_{k\alpha}|0\rangle,\ |4\rangle = c^{\dagger}_{k\beta}c^{\dagger}_{m\alpha}|0\rangle.$$

We compute the inner product between these four states and their duals and write the result as a 4×4 matrix:

	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$
$\langle 1\|$	1	q_2	$q_1 - q_2$	0
$\langle 2\|$	q_2	1	0	$q_1 - q_2$
$\langle 3\|$	$q_1 - q_2$	0	1	q_2
$\langle 4\|$	0	$q_1 - q_2$	q_2	1

The eigenvectors of this matrix are

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} \text{ and } \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}.$$

with corresponding eigenvalues $(1+q_1)$, $(1-q_1+2q_2)$, $(1-q_1)$ and $(1+q_1-2q_2)$ respectively. The form of these eigenvalues leads to the following consequences:

(a) All the four eigenvalues are positive inside the parallelogram BCFG depicted in figure 1. On each of the four boundaries of the parallelogram given by the straight lines: $q_1 = -1$, $q_2 = \frac{1}{2}(q_1 - 1)$, $q_1 = 1$ and $q_2 = \frac{1}{2}(q_1 + 1)$, one of the eigenvalues vanishes. Outside the parallelogram, one or more eigenvalues becomes negative. Hence the parallelogram demarcates the boundaries of the parameter space for which two particle vector space with positive definite metric exists.

(b) Since on each of the four sides BC, CF, FG and GB, one of the eigenvalues becomes zero, the corresponding eigenvector can be regarded as a null vector since

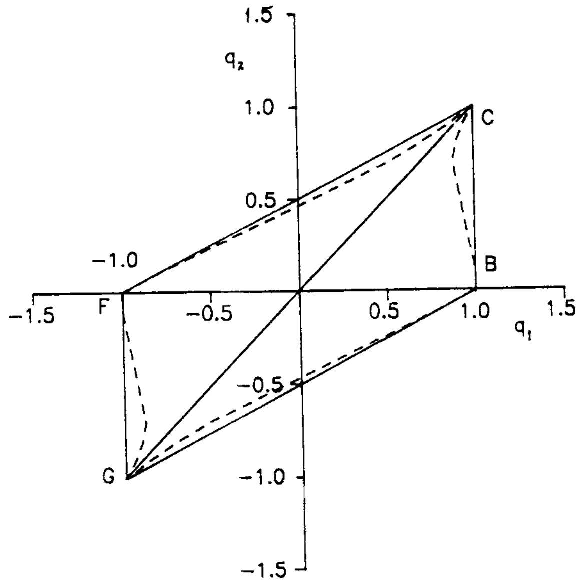

Figure 1. The parallelogram which bounds the allowed region in the parameter space for the two-particle sector. The four corners B, F, C and G correspond to Bose-Einstein, Fermi-Dirac, orthobose and orthofermi statistics respectively. The boundary of the allowed region for the three-particle sector with the spin index taking two values is shown as dashed line.

it has zero norm. These null vectors along FG, BC, BG and FC are respectively the following:

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c^\dagger_{k\alpha} + c^\dagger_{m\alpha}c^\dagger_{k\beta} + c^\dagger_{k\beta}c^\dagger_{m\alpha})|0\rangle = 0 \tag{3}$$

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} - c^\dagger_{m\beta}c^\dagger_{k\alpha} - c^\dagger_{m\alpha}c^\dagger_{k\beta} + c^\dagger_{k\beta}c^\dagger_{m\alpha})|0\rangle = 0 \tag{4}$$

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} - c^\dagger_{m\beta}c^\dagger_{k\alpha} + c^\dagger_{m\alpha}c^\dagger_{k\beta} - c^\dagger_{k\beta}c^\dagger_{m\alpha})|0\rangle = 0 \tag{5}$$

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c^\dagger_{k\alpha} - c^\dagger_{m\alpha}c^\dagger_{k\beta} - c^\dagger_{k\beta}c^\dagger_{m\alpha})|0\rangle = 0. \tag{6}$$

Thus, along each of the boundaries the vector space becomes three-dimensional.

(c) At each of the four corners B, C, F and G two of the eigenvalues vanish and so two eigenvectors become null vectors. Hence at the corners, the Hilbert space is reduced to a two-dimensional one.

Note that although there is no commutation rule on cc or $c^\dagger c^\dagger$ for any point inside the parallelogram, weaker forms of such rules (3), (4), (5) and (6) get generated on the four sides of the parallelogram, through the vanishing of the norms as explained above. At each of the four corners of the parallelogram, two such rules corresponding to the two sides meeting at that corner are simultaneously valid. Thus, for instance, at F, both (3) and (6) are valid, which can be combined to give

$$(c^\dagger_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c^\dagger_{k\alpha})|0\rangle = 0. \tag{7}$$

Further, (7) can be replaced by the operator identity

$$c^\dagger_{k\alpha}c^\dagger_{m\beta} + c^\dagger_{m\beta}c^\dagger_{k\alpha} = 0. \tag{8}$$

Similar things occur at all the four corners, thus leading to the strong commutation rules which we shall presently write down. We have however found that the replacement of eqs (3), (4), (5) and (6) by the corresponding operator identities is not possible and hence along the four sides, we have only the weaker conditions on the state vectors.

Let us now present the equations valid at the four corners of the parallelogram:

i) *At the corner F (Fermi-Dirac statistics)*

$$c_{k\alpha}c^{\dagger}_{m\beta} + c^{\dagger}_{m\beta}c_{k\alpha} = \delta_{km}\delta_{\alpha\beta} \tag{9}$$

$$c_{k\alpha}c_{m\beta} + c_{m\beta}c_{k\alpha} = 0 \tag{10}$$

ii) *At B (Bose-Einstein statistics)*

$$c_{k\alpha}c^{\dagger}_{m\beta} - c^{\dagger}_{m\beta}c_{k\alpha} = \delta_{km}\delta_{\alpha\beta} \tag{11}$$

$$c_{k\alpha}c_{m\beta} - c_{m\beta}c_{k\alpha} = 0 \tag{12}$$

iii) *At G (Orthofermi statistics)*

$$c_{k\alpha}c^{\dagger}_{m\beta} = \delta_{km}\delta_{\alpha\beta} - \delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{m\gamma}c_{k\gamma} \tag{13}$$

$$c_{k\alpha}c_{m\beta} + c_{m\alpha}c_{k\beta} = 0. \tag{14}$$

iv) *At C (Orthobose statistics)*

$$c_{k\alpha}c^{\dagger}_{m\beta} = \delta_{km}\delta_{\alpha\beta} + \delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{m\gamma}c_{k\gamma} \tag{15}$$

$$c_{k\alpha}c_{m\beta} - c_{m\alpha}c_{k\beta} = 0. \tag{16}$$

Orthofermi statistics was studied in an earlier paper (Mishra and Rajasekaran 1991) and it is characterized by a new exclusion principle which is more "exclusive" than Pauli's exclusion principle: an orbital state shall not contain more than one particle, whatever be the spin direction. Further, the wave function is antisymmetric in spatial indices alone, with the order of the spin-indices frozen. Both these properties follow from (14); the positions of α and β in this equation must be particularly noted. Orthobose statistics is the corresponding Bose-analogue; from (16), it follows that the wave function is symmetric in spatial indices alone, with the order of the spin-indices frozen.

Thus, at the four corners of the parallelogram, we have four kinds of statistics namely, Fermi-Dirac, Bose-Einstein, orthofermi and orthobose statistics which respectively correspond to total antisymmetry, total symmetry, spatial antisymmetry and spatial symmetry of the wave function.

One peculiarity of the new forms of statistics concerns the number operator. It was already pointed out in the earlier work (Mishra and Rajasekaran 1991) that, for orthofermi statistics, the total number operator for specific spatial index defined as

$$N_k = \sum_{\alpha} n_{k\alpha} = \sum_{\alpha} c^{\dagger}_{k\alpha}c_{k\alpha} \tag{17}$$

has the usual commutation relation with $c_{m\beta}$ but the number operator $n_{k\alpha}$ for specific spatial *and* spin index does not have the expected commutation relation with $c_{m\beta}$. Exactly the same situation prevails in the case of the orthobose statistics too. For general q_1 and q_2, number operators bilinear in c and $c^\dagger$ do not exist. At $q_1 = q_2 = 0$, the number operator can be expressed as an infinite series in powers of c and $c^\dagger$ (Greenberg 1990). For $q_2 = 0$ and for arbitrary q_1 the form of this infinite series is known only for the case of a single mode system (Chaturvedi *et al* 1991; Chakrabarti and Jagannathan 1991).

We now come to the diagonals. Along the diagonal BF, $(q_2 = 0)$, the algebra (1) reduces to the q-mutator algebra of Greenberg (1991):

$$c_{k\alpha}^\dagger c_{m\beta}^\dagger - q_1 c_{m\beta}^\dagger c_{k\alpha} = \delta_{km}\delta_{\alpha\beta} \tag{18}$$

for which Fivel (1990) and Zagier (as quoted by Greenberg (1991)) have already proved the positive-definiteness of the metric for the n-particle sector in the region $-1 \leqslant q_1 \leqslant 1$. These are '$q$-ons' having orbital and spin indices.

Along the other diagonal CG $(q_1 = q_2)$, we have the algebra:

$$c_{k\alpha} c_{m\beta}^\dagger - q_2 \delta_{\alpha\beta} \sum_\gamma c_{m\gamma}^\dagger c_{k\gamma} = \delta_{km}\delta_{\alpha\beta}. \tag{19}$$

In this case, the inner product in the n-particle sector is

$$\langle 0|\ldots c_{t\gamma} c_{m\beta} c_{k\alpha} c_{p\lambda}^\dagger c_{s\mu}^\dagger c_{q\nu}^\dagger \ldots |0\rangle = q_2^P \delta_{\alpha\lambda}\delta_{\beta\mu}\delta_{\gamma\nu}\ldots \tag{20}$$

where $(p, s, q\ldots)$ is some permutation of $(k, m, t\ldots)$ and P is the number of inversions* in this permutation of orbital indices. We see that the spin-ordering is frozen and as for the orbital indices the behaviour is exactly that of q-ons possessing only orbital index. Hence, the Fivel-Zagier proof of positivity of the metric for arbitrary n-particle states is applicable along this diagonal also, for $-1 \leqslant q_2 \leqslant 1$. This algebra (19) interpolates between orthofermi and orthobose statistics.

The result (20) also suggests the possibility of factorization between the orbital and spin indices. We assume

$$c_{k\alpha} = f_k b_\alpha \tag{21}$$

$$f_k b_\alpha - b_\alpha f_k = f_k^\dagger b_\alpha - b_\alpha f_k^\dagger = 0 \tag{22}$$

$$f_m f_k^\dagger - q_2 f_k^\dagger f_m = \delta_{km} \tag{23}$$

and ask what is the algebra satisfied by b_α if we impose the algebra (19) on $c_{k\alpha}$. The answer is the following:

$$b_\alpha b_\beta^\dagger = \delta_{\alpha\beta} \tag{24}$$

$$\sum_\gamma b_\gamma^\dagger b_\gamma = 1. \tag{25}$$

Equations (24) and (25) define an algebra originally studied by Cuntz (1977). Thus, on the diagonal CG, the factorization ansatz leads to an elegant decomposition of $c_{k\alpha}$ into q-mutator algebra in orbital indices and Cuntz algebra in the spin indices.

* Number of inversions is defined as the minimum number of transpositions of successive indices necessary to bring $(k, m, t\ldots)$ into $(p, s, q,\ldots)$ (Greenberg 1991).

What about the positivity of the inner product in the two-dimensional region? Do the sides of the parallelogram continue to be the boundaries of the region of positivity for three and more particles? We have found that the answer is in the negative. The relevant calculations become much more complex since the total number of states increases rapidly. The details will be presented in a longer paper, but the result for three particles (with spin indices taking only two values) is shown in figure 1. Thus the region of positivity shrinks. How does the boundary of this region move as the number of particles n increases further and what is the limiting boundary for $n \to \infty$? These are open questions for the present. However, we know that complete Hilbert spaces with positive-definite metric exist along the two diagonals and the final answers must be consistent with this fact.

Finally we may mention that the algebra described by eq. (1) is invariant under the unitary transformations on the space indices:

$$d_{k\alpha} = \sum_p U_{kp} c_{p\alpha}; \quad U^\dagger U = U U^\dagger = 1 \tag{26}$$

as well as similar unitary transformations on the spin indices:

$$e_{k\alpha} = \sum_\lambda V_{\alpha\lambda} c_{k\lambda}; \quad V^\dagger V = V V^\dagger = 1. \tag{27}$$

Invariance under such unitary transformations is an important requirement on a quantum system in the general context as we have already discussed earlier (Mishra and Rajasekaran 1991). It is this requirement which dictates the form of the term $q_2 \delta_{\alpha\beta} \Sigma_\gamma c^\dagger_{m\gamma} c_{k\gamma}$ in eq. (1). Further, it is easily seen that more terms such as $q_3 \delta_{km} \Sigma_p \cdot c^\dagger_{p\beta} c_{p\alpha}$ and $q_4 \delta_{km} \delta_{\alpha\beta} \Sigma_{p,\gamma} c^\dagger_{p\gamma} c_{p\gamma}$ can be added to eq. (1) and the equation will still be invariant under the above unitary transformations. These more general algebras as well as related issues will be discussed elsewhere.

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