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Field localization and particle confinement effects on pair creation by Schwinger's mechanism

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Abstract. Solutions of the Dirac equation in the presence of a static uniform electric field ε in the z-direction and a linear confining potential Az, are obtained. Generalized reflection and transmission coefficients are derived for such divergent potentials for $\varepsilon > A/e$. The eigen-spectrum and corresponding localized eigenfunctions for $\varepsilon < A/e$ are obtained from the reflection coefficient and the continuum solutions respectively. The rate for the electric field to decay into pairs is derived from the transmission coefficient. Neglecting nonabelian effects in quantum chromodynamics we identify the field ε with a colour electric field and the produced particles with a quark and an antiquark. By considering a cylindrical geometry, we thus obtain a generalization of Schwinger's formula, for the field ε in a finite spatial region with the quark (antiquark) being confined in the z direction by the linear potential Az and in the perpendicular direction by the MIT bag boundary condition. The result is used to qualitatively study Schwinger's mechanism of quark-gluon plasma (QGP) formation in ultrarelativistic heavy ion collisions. It is found that the critical strength of the field required to create $q\bar{q}$ pairs is enhanced, $\varepsilon_c(A) > \varepsilon_c(A=0)$. The rate of pair creation for constant ε, decreases for non-zero A, implying longer QGP formation times. Because of $\varepsilon_c(A) > \varepsilon_c(0)$, QGP is predicted to be formed in the early stages of the nuclear collision. The finite size effects and the MIT bag boundary condition effects on QGP formation are also discussed.

Keywords. Pair creation in electric field; confinement; reflection and transmission coefficients; quark gluon plasma.

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1. Introduction

In ultrarelativistic heavy ion collisions it is believed that a large amount of energy is deposited in a small space-time region. It is this energy stored in the form of colour electric field energy which is thought to be responsible for the production of quark-antiquark ($q\bar{q}$) pairs and eventually a quark-gluon plasma (QGP). The $q\bar{q}$ pairs could be produced by either of the following two mechanisms (i) the gluon induced pair creation in the presence of a strong colour electric field ε, and (ii) the spontaneous breakdown of the static field ε into $q\bar{q}$ pairs as in the interpretation of the Klein paradox. The latter mechanism of producing QGP is considered here. In the flux tube model of heavy ion collision, the ε field configuration is assumed to be formed in a cylindrical region joining the two nuclei receding from each other after undergoing a transparent collision. The two Lorentz contracted nuclei are treated as two colour charged capacitor plates. The radius of the flux tube connecting the plates is of the order of 1 fm. The collision leads to a configuration of several tubes with different colour charges at their ends. In order to produce QGP by a spontaneous breakdown of ε it is essential that the produced q and $\bar{q}$ remain confined to this region. Heisenberg and Euler (1936) and Schwinger (1951) have proposed a mechanism for pair creation by breakdown of an electric field. Since then many alternate derivations of their formula have been proposed (Daugherty and Lerche 1976; Dosch and Gromes 1986; Cox and Yidiz 1985; Glendenning and Matsui 1983). Most of these have field in an infinite volume and no confinement. Some recent works Wang and Wong (1988), Martin and Vautherin (1989) generalized this to the localized field configuration but with no confinement of $q\bar{q}$. Schönfeld *et al* (1990) considered a cylindrical region of space and imposed confinement in the radial direction by means of the MIT bag boundary condition; there was no confinement in the z-direction.

The aim of the present paper is to generalize Schwinger's formula for the pair production rate when the ε field configuration is in the shape of an infinitely long cylinder of finite radius, and when the pair is confined by a linear confining potential along the length of the cylinder and by the MIT bag confining condition in the radial direction. The derivation uses the generalized scattering formalism of the problem. The localized solutions are also studied using the results of the scattering formalism. Implications of these results to QGP formation are studied qualitatively.

2. Generalized scattering solution: reflection and transmission coefficients

The flux tube is assumed to have a constant uniform colour electric field ε along its axis (taken as the z-axis) in the same manner as the constant uniform electric field between two condenser plates. The colour charge in analogy with QED is denoted by e, with the assumption of an abelian field breaking into $q\bar{q}$ pairs under confining potentials. The vector-coupling potential due to this field is $A^\mu = (0, 0, 0; \phi)$, $\phi(z) = e\varepsilon z$. The scalar-coupling potential is taken as $A(\rho)z$ which makes the $q(\bar{q})$ mass z and ρ dependent: $m(\rho, z) = m + A(\rho)z$. $A(\rho)$ is a scalar colour-dielectric constant which is taken as constant A for $\rho < R$ (the radius of the flux tube) and zero on its cylindrical surface. With this the Dirac equation in the cylindrical tube region is

$$[\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m(z) - (E - \phi)]\Psi = 0. \tag{1}$$

(The Bjorken and Drell convention of Dirac matrices γ^μ and $\boldsymbol{\alpha}, \beta$ is used, and c and $\hbar$ are set equal to 1.) In (1), we have considered a combination of scalar and vector coupling potentials. It is known (Greiner *et al* 1985) that, in contrast to vector coupling, the scalar coupling cannot spontaneously create pairs, no matter how strong the potential is. The confinement in the radial direction is achieved by means of the MIT bag boundary condition on the cylindrical surface

$$(1 + i\boldsymbol{\gamma}\cdot\hat{\rho})\Psi = 0, \tag{2}$$

where $\hat{\rho}$ is the unit vector perpendicular to the cylindrical surface. This requirement quantizes the transverse motion in the flux tube. It is important to note that just having ε in a finite region does not confine quarks. Their confinement requires independent external fields or boundary conditions. Confinement leads to $q(\bar{q})$ probabilities at infinite separation zero; at the same time, for $q\bar{q}$ pair creation, $q(\bar{q})$ separation by large distances is essential. The presence of a confinement potential, therefore, requires a generalization of the continuum solutions of the Dirac equation.

Let us substitute

$$\Psi = [\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m(z) + E - \phi]\beta\Phi, \tag{3}$$

in (1) to get the following equation for the spinor Φ,

$$[\mathbf{p}_t^2 + p_3^2 + m^2(z) - (E-\phi)^2 - i(\alpha_3 e\varepsilon + \gamma^3 A)]\Phi = 0. \tag{4}$$

The spinor part of Φ in (4), Φ_0, can be found easily by diagonalizing the constant matrix [the last term in (4)]

$$(\alpha_3 e\varepsilon + \gamma^3 A)\Phi_0 = \bar{\lambda}\Phi_0. \tag{5}$$

The four independent solutions of (5) are

$$\Phi_0 = \begin{pmatrix} (e\varepsilon + A)^{1/2}\chi^s \\ k(e\varepsilon - A)^{1/2}\chi^s \end{pmatrix}, \tag{6}$$

corresponding to the eigenvalues

$$\bar{\lambda} = ks(e^2\varepsilon^2 - A^2)^{1/2} \equiv ks\lambda. \tag{7}$$

The χ^s are the usual two-component Pauli spinors satisfying $\sigma_3\chi^s = s\chi^s$. The values of k and s are ± 1. Because of the cylindrical symmetry of the problem, the general solution of (4) is of the form

$$\Phi = \exp(i\mu\varphi) J_\mu(\kappa\rho) f(z)\Phi_0(k,s). \tag{8}$$

The equation for $f(z)$ follows immediately from (4), (5), (8) and from the following relation,

$$\mathbf{p}_t^2[\exp(i\mu\varphi) J_\mu(\kappa\rho)] = \kappa^2[\exp(i\mu\varphi) J_\mu(\kappa\rho)].$$

Thus

$$\frac{d^2 f}{dz^2} + [(E - e\varepsilon z)^2 - \kappa^2 - m^2(z) + i\lambda ks]f = 0. \tag{9}$$

If we define

$$\eta = (e^2\varepsilon^2 - A^2)^{1/4}[z - (e\varepsilon E + mA)/(e^2\varepsilon^2 - A^2)], \quad \omega = (1-i)\eta,$$
$$\beta = -[\kappa^2 + (me\varepsilon + EA)^2/(e^2\varepsilon^2 - A^2)]/(e^2\varepsilon^2 - A^2)^{1/2}; \tag{10}$$

eq. (9) acquires the following simple form

$$\frac{d^2 f}{d\omega^2} + \left[\left(\frac{i\beta}{2} - \frac{ks}{2}\right) - \frac{\omega^2}{4}\right] f = 0. \tag{11}$$

In the above equations and in what follows in this section we assume $\varepsilon > A/e$. Solutions of (11) are

$$f(\omega) = D_{\nu_-}(\omega) \text{ for } ks = 1,\ \nu_- = i\beta/2 - 1,$$
$$f(\omega) = D_{\nu_+}(\omega) \text{ for } ks = -1,\ \nu_+ = i\beta/2, \tag{12}$$

where D's are the parabolic cylinder functions (Gradshteyn and Ryzhik 1965). Therefore the solutions in (8) are

$$\Phi(\kappa,\mu,k,s)=\exp(i\mu\varphi)J_\mu(\kappa\rho)D_{v\pm}(\omega)\Phi_0(k,s). \quad (13)$$

Since (11) is a second-order differential equation, all the Ψ's obtained from (3) and (13) need not be the solutions of (1) which is a linear differential equation. The requirement that Ψ should satisfy (1) fixes the value of k vis-à-vis s, so that there are only four solutions corresponding to four different values of s and $v, s=\pm 1$ and $v=v_\pm$. In deriving the expression of Ψ from (3) and (13) we use

$$\begin{aligned}&\boldsymbol{\sigma}\cdot\mathbf{p}_t\chi^s=(p_1+isp_2)\chi^{-s},\\&(p_1+isp_2)[\exp(i\mu\varphi)J_\mu(\kappa\rho)]=i\kappa s\exp i(\mu+s)\varphi J_{\mu+s}(\kappa\rho),\end{aligned} \quad (14)$$

and the fact that Ψ depends on ω only through D_v. We introduce the following notation in order to simplify the expression for the wave function Ψ:

$$\begin{aligned}&g_{\kappa,\mu}(\rho,\varphi)=\exp(i\mu\varphi)J_\mu(\kappa\rho),\ D_v^i=D_{v_i}(\omega),\ c^+=v=i\beta/2,\ c^-=1,\\&f(A)=-(e^2\varepsilon^2-A^2)^{1/4},\ v_i=(v_+,v_-),\ \alpha_1(A)=(1+i)(e\varepsilon-A)^{1/2},\\&\alpha_2(A)=[(e\varepsilon+A)/(e\varepsilon-A)]^{1/4}(me\varepsilon+EA)/(e\varepsilon+A),\\&\alpha_3(A)=\kappa[(e\varepsilon-A)/(e\varepsilon+A)]^{1/4},\ \beta_j(A)=\alpha_j(-A).\end{aligned} \quad (15)$$

In terms of these variables the solutions Ψ obtained from (3) and (13) are [in the following, $i=(+,-)$]

$$\Psi^i(\kappa,\mu,v,s)=f(A)\begin{bmatrix}(\alpha_1c^iD_v^{-i}-\alpha_2D_v^i)g_{\kappa,\mu}\chi^s\mp i\alpha_3D_v^ig_{\kappa,\mu+s}\chi^{-s}\\ \pm s(\beta_1c^iD_v^{-i}+\beta_2D_v^i)g_{\kappa,\mu}\chi^s-is\beta_3D_v^ig_{\kappa,\mu+s}\chi^{-s}\end{bmatrix}. \quad (16)$$

The generalized reflection (R) and transmission (T) coefficients are obtained from asymptotic form of Ψ in (16). Using the asymptotic expansions of $D_v(\omega)$ (Gradshteyn and Ryzhik 1965) the asymptotic forms of $\Psi^\pm$ can be written down. To this end let us introduce normalized spinors ϕ_i and the phase function $\phi(\eta)$

$$\begin{aligned}&\phi_0=\left\{\begin{pmatrix}-\alpha_2\\ \pm s\beta_2\end{pmatrix}g_{\kappa,\mu}\chi^s-i\begin{pmatrix}\alpha_3\\ s\beta_3\end{pmatrix}g_{\kappa,\mu+s}\chi^{-s}\right\}\Big/N_0,\\&\phi_1=\begin{pmatrix}\alpha_1\\ \pm s\beta_1\end{pmatrix}g_{\kappa,\mu}\chi^s/N_1,\\&\phi(\eta)=\eta^2/2+\beta/2\log(\sqrt{2}|\eta|),\\&N_0^2=2e\varepsilon[\kappa^2+(me\varepsilon+EA)^2/(e^2\varepsilon^2-A^2)]/(e^2\varepsilon^2-A^2)^{1/2},\\&N_1^2=4e\varepsilon,\ N_1^{+2}=e\varepsilon\beta^2.\end{aligned} \quad (17)$$

Then the asymptotic forms of Ψ are as follows:

$$\begin{aligned}&\Psi^+(z:\infty)=f(A)N_0\exp(\pi\beta/8)\exp[i\phi(\eta)]\phi_0,\\&\Psi^-(z:\infty)=f(A)N_1\exp(\pi\beta/8)\exp[i\phi(\eta)]\phi_1,\end{aligned}$$

$$\Psi^{+}(z:-\infty)=f(A)N_0\exp(-3\pi\beta/8)\exp[i\phi(\eta)]\phi_0$$
$$-f(A)N_1^{+}[\sqrt{2\pi}/\Gamma(1-i\beta/2)]\exp(-\pi\beta/8)\exp[-i\phi(\eta)]\phi_1,$$
$$\Psi^{-}(z:-\infty)=f(A)N_1\exp(-3\pi\beta/8)\exp[i\phi(\eta)]\phi_1$$
$$-f(A)N_0[\sqrt{2\pi}/\Gamma(1-i\beta/2)]\exp(-\pi\beta/8)\exp[-i\phi(\eta)]\phi_0. \quad (18)$$

From (18), the reflection and the transmission coefficients can be defined as follows

$$\begin{aligned}R_- &= |N_0[\sqrt{2\pi}/\Gamma(1-i\beta/2)]/N_1|^2\exp[\pi\beta/2]\\ &= 2\exp[\pi\beta/2]\sinh(\pi\beta/2)[\kappa^2+(me\varepsilon+EA)^2/(e^2\varepsilon^2-A^2)]\\ &\quad\times(e^2\varepsilon^2-A^2)^{-1/2}/\beta\\ &= -2\exp[\pi\beta/2]\sinh(\pi\beta/2)=-(\exp[\pi\beta]-1)\\ T_- &= \exp[\pi\beta],\\ R_+ &= |N_1^{+}[\sqrt{2\pi}/\Gamma(1-i\beta/2)]/N_0|^2\exp[\pi\beta/2]\\ &= -2\exp[\pi\beta/2]\sinh(\pi\beta/2),\\ T_+ &= \exp[\pi\beta]. \end{aligned}\quad (19)$$

It is seen that $R_+ = R_- = R$ and $T_+ = T_- = T$ and $R + T = 1$ as it should. The transmission coefficient is given by

$$T=\exp[-\pi\{\kappa^2+(me\varepsilon+EA)^2/(e^2\varepsilon^2-A^2)\}/(e^2\varepsilon^2-A^2)^{1/2}]. \quad (20)$$

3. Bound state solutions and reflection coefficient

In § 2, the scattering solutions were studied for $\varepsilon > A/e$. If, on the other hand, $\varepsilon < A/e$ then bound state solutions are expected. In this case the phase function $\phi(\eta)$ should become an imaginary quantity so that the decaying solutions Ψ can be found. We have

$$\begin{aligned}\eta &= (e^2\varepsilon^2-A^2)^{1/4}[z-(e\varepsilon E+mA)/(e^2\varepsilon^2-A^2)]\\ &= \sqrt{i}(A^2-e^2\varepsilon^2)^{1/4}[z+(e\varepsilon E+mA)/(A^2-e^2\varepsilon^2)]\equiv\sqrt{i}\,x.\end{aligned}\quad (21)$$

Therefore the phase of the asymptotic solutions in (18) becomes $\exp[i\phi(\eta)]\simeq\exp(-x^2/2)$, as required. However, the other term $\exp[-i\phi(\eta)]$ would diverge. Therefore, as in a scattering problem, we demand that the reflection coefficient must vanish for the bound state solutions to exist, i.e., the coefficient of the diverging term $\exp[x^2/2]$ is zero. This gives

$$R = 1-\exp(\pi\beta)=0. \quad (22)$$

From (10) if $\varepsilon < A/e$, β becomes

$$\beta = i[\kappa^2-(me\varepsilon+EA)^2/(A^2-e^2\varepsilon^2)]/(A^2-e^2\varepsilon^2)^{1/2}. \quad (23)$$

From (22) and (23), the condition for the existence of bound states becomes

$$[\kappa^2 - (me\varepsilon + EA)^2/(A^2 - e^2\varepsilon^2)]/(A^2 - e^2\varepsilon^2)^{1/2} = -2n, \tag{24}$$

where n is an integer $0, \pm 1, \ldots$ etc. The bound state discrete eigenvalues are the solutions of the quadratic eq. (24) and they are

$$E_n = -me\varepsilon/A \pm [2n(A^2 - e^2\varepsilon^2)^{3/2} + \kappa^2(A^2 - e^2\varepsilon^2)]^{1/2}/A. \tag{25}$$

The wave functions corresponding to these energies are also not difficult to derive from the known solutions in (16). Towards this end let us first notice from (10) and (21) that

$$\omega = (1-i)\eta = (1-i)\sqrt{i}\,x = \sqrt{2}x. \tag{26}$$

From (23) and (24)

$$\beta = -2ni,\ v_+ = i\beta/2 = n,\ v_- = i\beta/2 - 1 = n - 1. \tag{27}$$

Therefore, from (26) and (27) [recall $i = (+, -)$]

$$\begin{aligned} D_v^i(\omega) &= D_{v_i}(\omega) = [D_n(\sqrt{2}\,x), D_{n-1}(\sqrt{2}\,x)] \\ &= 2^{-n/2}[\exp(-x^2/2)H_n(x), \sqrt{2}\exp(-x^2/2)H_{n-1}(x)] \\ &= D_n^i. \end{aligned} \tag{28}$$

Similarly the constants in (15) can be analytically continued in the region $\varepsilon < A/e$ to obtain

$$\begin{aligned} & c^+ = i\beta/2 = n = d^+, c^- = d^- = 1, \\ & \alpha_1(A) = i(1+i)\sqrt{A - e\varepsilon} = -\sqrt{2}(i)^{-1/2}a_1(A), \\ & \alpha_2(A) = (i)^{-1/2}[(A+e\varepsilon)/(A-e\varepsilon)]^{1/4}(me\varepsilon + EA)/(e\varepsilon + A) = (i)^{-1/2}a_2(A), \\ & \alpha_3(A) = (i)^{1/2}\kappa[(A-e\varepsilon)/(A+e\varepsilon)]^{1/4} = (i)^{1/2}a_3(A), \\ & c^i\beta_1 = \sqrt{2}\sqrt{i}(e\varepsilon + A)^{1/2}d^i = \sqrt{2}\sqrt{i}\,d^i b_1(A), c^i = (c^+, c^-), \text{ etc.}, \\ & f(A) = -\sqrt{i}(A^2 - e^2\varepsilon^2)^{1/4} = -\sqrt{i}\,f_0(A), \\ & \beta_2 = -\sqrt{i}[(A-e\varepsilon)/(A+e\varepsilon)]^{1/4}(me\varepsilon - EA)/(A - e\varepsilon) = -\sqrt{i}b_2(A), \\ & \beta_3 = (\kappa/\sqrt{i})[(A+e\varepsilon)/(A-e\varepsilon)]^{1/4} = b_3(A)/\sqrt{i}. \end{aligned} \tag{29}$$

Using these relations in (16) the following wave functions are obtained.

$$\Psi^i(\kappa, \mu, n, s) = 2^{-ni/2} f_0(A) \begin{bmatrix} (a_1 d^i E_n^{-i} + a_2 E_n^i) g_{\kappa,\mu}\chi^s \mp a_3 E_n^i g_{\kappa,\mu+s}\chi^{-s} \\ -is(b_1 d^i E_n^{-i} \mp b_2 E_n^i) g_{\kappa,\mu}\chi^s - b_3 E_n^i g_{\kappa,\mu+s}\chi^{-s} \end{bmatrix}, \tag{30}$$

where $n_\pm = (n, n-1)$ and $E_n^i = H_{ni}(x)\exp(-x^2/2)$. These solutions are found to be identical to the bound state solutions obtained directly from the Dirac equation when

$\varepsilon < A/e$ (Ru-keng and Yuhong 1984). Naturally, in this situation pairs cannot be created as the solutions vanish at large separations.

4. The MIT bag boundary condition

For the boundary condition in (2), the solutions on the cylindrical surface $\rho = R$ where the scalar potential constant $A(\rho = R)$ vanishes, are required. These are derived from (16) by substituting $A = 0$:

$$\Psi^i = -\sqrt{e\varepsilon}\begin{pmatrix} [\sqrt{e\varepsilon}(1+i)c^i D_v^{-i} - mD_v^i]g_{\kappa,\mu}\chi^s \mp i\kappa D_v^i g_{\kappa,\mu+s}\chi^{-s} \\ \pm s[\sqrt{e\varepsilon}(1+i)c^i D_v^{-i} + mD_v^i]g_{\kappa,\mu}\chi^s - is\kappa D_v^i g_{\kappa,\mu+s}\chi^{-s} \end{pmatrix}. \tag{31}$$

The operator $i\gamma\cdot\hat{\rho}$ acting on a wave function of a given spin projection, say $s = 1$, (denoted by Ψ_1) mixes up this wave function with that corresponding to $s = -1$ (denoted by Ψ_2), whereas it does not alter the other quantum numbers κ, μ and v. Therefore, the quark wave function compatible with this boundary condition is a superposition of $s = 1$ and $s = -1$ solutions of the same values of κ, μ and v. Let the required combination be

$$\Psi = a\Psi_1 + b\Psi_2 = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}.$$

The boundary condition in (2) leads to the following two equations for the two-component spinors Φ_1 and Φ_2, the solutions of which determine the coefficients a and b.

$$\Phi_1 + i\sigma\cdot\hat{\rho}\Phi_2 = 0; \; \Phi_2 - i\sigma\cdot\hat{\rho}\Phi_1 = 0.$$

It may be seen that the second equation can be obtained from the first by multiplying it with $-i\sigma\cdot\hat{\rho}$ from the left. Hence it is identically satisfied if the first is satisfied. Thus the two independent equations to determine a and b are given by

$$\Phi_1 + i\sigma\cdot\hat{\rho}\Phi_2 = 0. \tag{32}$$

Substituting the solutions (31) in (32) one obtains

$$D_v[am + b\sqrt{e\varepsilon}(1+i)]J_\mu + D_{v-1}[bm - a\sqrt{e\varepsilon}(1+i)v]J_\mu - \kappa[aD_v + bD_{v-1}]J_{\mu+1} = 0, \tag{33}$$

$$\kappa[aD_v - bD_{v-1}]J_{\mu+1} - D_v[am + b\sqrt{e\varepsilon}(1+i)]J_\mu + D_{v-1}[bm - a\sqrt{e\varepsilon}(1+i)v]J_\mu = 0. \tag{34}$$

Since D_v and D_{v-1} are independent functions of z and since the above equations are satisfied for all values of z, the following equations follow from (33).

$$a(mJ_\mu - \kappa J_{\mu+1}) + b\sqrt{e\varepsilon}(1+i)J_\mu = 0,$$

$$a\sqrt{e\varepsilon}(1+i)vJ_\mu - b(mJ_\mu - \kappa J_{\mu+1}) = 0. \tag{35}$$

Other two equations that follow from (34) are identical with those in (35). The solution is

$$a/b = \sqrt{e\varepsilon}(1+i)J_\mu/(\kappa J_{\mu+1} - mJ_\mu),$$

and

$$\kappa(J_{\mu+1}^2 - J_\mu^2) - 2mJ_\mu J_{\mu+1} = 0. \tag{36}$$

In the derivation of (36), we have used $v = -i\beta/2 = i(\kappa^2 + m^2)/2e\varepsilon$, see (10). In (36) $J_\mu = J_\mu(\kappa R)$. It quantizes the transverse motion of $q(\bar{q})$ in the flux tube. The discrete values of κ, the solutions of (36), are the only ones allowed in the wave functions. These wave functions thus include effects of both the finite spatial extension of the ε field and the confinement condition on quarks in the tube.

5. Pair creation rate and generalization of Schwinger's formula

In this section the transmission coefficient is used to calculate the vacuum to $q\bar{q}$ pair transition rate. The approach developed by Casher *et al* (1979) is used. In their approach it is assumed that the pair creation takes place via an intermediate excited state of the vacuum. The virtual pair is created instantaneously at spatial points $(z, \mathbf{\rho})$ with total energy $E_1 = E_2 = 0$ and the probability of creation of this state is unity. Energy is conserved between the initial vacuum state and the final pair state; however, the intermediate states need not have the same energy. The transition from virtual to physical pair state occurs via tunneling. It is assumed that the individual pair creation events are uncorrelated. In our work the state of a $q(\bar{q})$ is specified by κ, μ, spin (s), isospin (τ) and its longitudinal momentum p_3 given by the classical equation $\mathrm{d}p_3/\mathrm{d}t = e\varepsilon_{\mathrm{eff}}$. From the above assumption of instantaneous pair creation, and (20), the probability of creating a pair, $P(\kappa_n, \mu, p_3)$, is given by

$$P(\kappa_n, \mu, p_3) = T = \exp[-\pi\{\kappa_n^2 + m^2 e^2\varepsilon^2/(e^2\varepsilon^2 - A^2)\}/(e^2\varepsilon^2 - A^2)^{1/2}], \tag{37}$$

where κ_n is one of the solutions of (36). In (37) we have dropped the spin, isospin indices and used the fact that the quantum numbers of q and $\bar{q}$ are equal and opposite for $q\bar{q}$ creation from vacuum. In terms of this, the vacuum persistence probability is (Casher *et al.* 1979).

$$|\langle 0+|0-\rangle|^2 = \prod_{s,\tau}\prod_{p_3}\prod_{n}\prod_{\mu}[1 - P(\kappa_n, \mu, p_3)]$$
$$= \exp\left\{\sum_{s,\tau}\sum_{p_3}\sum_{n,\mu}\log[1 - P(\kappa_n, \mu, p_3)]\right\}. \tag{38}$$

In the presence of a constant electric field ε along the z-axis, the wave function f satisfies (9). If $A = 0$, (9) can be written as

$$(p_z^2 + \kappa^2 + m^2 - ie\varepsilon ks)f = (E - e\varepsilon z)^2 f. \tag{39}$$

By analogy with the Klein-Gordon equation, we identify $(E - e\varepsilon z)$ with the effective energy E_{eff}, and write the effective classical force $F = e\varepsilon$ as

$$F = -\partial E_{\text{eff}}/\partial z. \tag{40}$$

The imaginary term on the left-hand side of (39) reflects the loss of flux from the initial state (decay of the vacuum). On the other hand, if $A \neq 0$, (9) can be written as

$$[p_z^2 + \kappa^2 + m_{\text{eff}}^2 - i(e^2\varepsilon^2 - A^2)^{1/2}ks]f = E_{\text{eff}}^2 f,$$

where

$$m_{\text{eff}}^2 \equiv (me\varepsilon + EA)^2/(e^2\varepsilon^2 - A^2),$$

and

$$E_{\text{eff}} \equiv (e^2\varepsilon^2 - A^2)^{1/2}[(Ee\varepsilon + mA)/(e^2\varepsilon^2 - A^2) - z].$$

If the prescription (40) is used in this case, the effective classical force F in the presence of a confining potential would be

$$F = (e^2\varepsilon^2 - A^2)^{1/2},$$

and the effective classical equation of motion of this particle would be

$$\frac{\mathrm{d}p_3}{\mathrm{d}t} = (e^2\varepsilon^2 - A^2)^{1/2}. \tag{41}$$

Since P in (38) is independent of p_3, the sum in the exponential function can now be performed. Let L be the length of the flux tube (very large but finite) and T the total time for the vacuum to decay into pairs. Using (41), it follows that

$$\sum_{p_3} 1 = \frac{L}{2\pi}\int \mathrm{d}p_3 = \frac{L}{2\pi}\int \frac{\mathrm{d}p_3}{\mathrm{d}t}\,\mathrm{d}t = \frac{L}{2\pi}\,T(e^2\varepsilon^2 - A^2)^{1/2}. \tag{42}$$

The probability P per unit volume per unit time to produce pairs in the flux tube is defined in terms of the vacuum persistence probability by the relation

$$P = -[(e^2\varepsilon^2 - A^2)^{1/2}\gamma/2\pi^2R^2]\sum_{n,\mu}\log[1 - P(\kappa_n, \mu)], \tag{43}$$

where γ is the spin-isospin factor. Expanding the logarithm in (43) and using (37),

$$P = [(e^2\varepsilon^2 - A^2)^{1/2}\gamma/2\pi^2R^2]\sum_N (1/N)\exp\left[\frac{-N\pi m^{*2}}{(e^2\varepsilon^2 - A^2)^{1/2}}\right] \times \sum_{n,\mu}\exp\left[\frac{-N\pi\kappa_n^2}{(e^2\varepsilon^2 - A^2)^{1/2}}\right]. \tag{44}$$

In (44), the effective mass m^* is defined as follows

$$m^* = m/(1 - A^2/e^2\varepsilon^2)^{1/2} > m$$

since $A^2/e^2\varepsilon^2 < 1$.

In the limit of an infinite volume, let us evaluate the sum (the area πR^2 will be replaced by ℓ^2)

$$\begin{aligned}
&\frac{1}{\pi R^2}\sum_{n,\mu}\exp[-N\pi\kappa_n^2/(e^2\varepsilon^2-A^2)^{1/2}]\\
&=\frac{1}{\ell^2}\left(\frac{\ell}{2\pi}\right)^2\int \mathrm{d}\boldsymbol{\kappa}\exp[-N\pi\kappa^2/(e^2\varepsilon^2-A^2)^{1/2}]\\
&=(e^2\varepsilon^2-A^2)^{1/2}/4\pi^2 N.
\end{aligned}\tag{45}$$

Substituting it in (44)

$$P=[\gamma(e^2\varepsilon^2-A^2)/8\pi^3]\sum_N(1/N^2)\exp[-N\pi m^{*2}/(e^2\varepsilon^2-A^2)^{1/2}].\tag{46}$$

Even if the prescription (40) is not accurate enough, the above expression for P is still correct except for the factor $(e^2\varepsilon^2-A^2)$ outside the summation. This is the generalization of Schwinger's formula to the case of an infinite region of colour electric field with the confinement of quarks in the z-direction. As the probability is per unit volume per unit time, it has dimensions of $e^2\varepsilon^2$. Written in a dimensionless form

$$\begin{aligned}
p(A)&=P/e^2\varepsilon^2\\
&=[\gamma(m/m^*)^2/8\pi^3]\sum_N(1/N^2)\exp[-N\pi m^{*2}/(e^2\varepsilon^2-A^2)^{1/2}].
\end{aligned}\tag{47}$$

Since the effective mass $m^*(A)>m$, the probability of pair creation decreases in the presence of a confining potential, for the same strength of ε. The rate of $q\bar{q}$ pair creation reduces to Schwinger's formula if $A=0$. In order to study the finite size effect on the pair creation rate, (44) is rewritten as

$$P=\left[\frac{(e^2\varepsilon^2-A^2)^{1/2}\gamma}{2\pi}\right]\sum_N N^{-2}\exp\left[\frac{-N\pi m^{*2}}{(e^2\varepsilon^2-A^2)^{1/2}}\right]\{S_N(R:\infty)+\Delta S_N(R)\},\tag{48}$$

here

$$S_N(R)=(N/\pi R^2)\sum_{n,\mu}\exp[-N\pi\kappa_n^2/(e^2\varepsilon^2-A^2)^{1/2}],$$

$$\Delta S_N(R)=S_N(R)-S_N(R:\infty).$$

Using the result in (45) we get

$$S_N(R:\infty)=(e^2\varepsilon^2-A^2)^{1/2}/4\pi^2.$$

Substituting this in (48) we obtain

$$\begin{aligned}
P=&\left[\frac{\gamma(e^2\varepsilon^2-A^2)}{8\pi^3}\right]\sum_N N^{-2}\exp\left[\frac{-N\pi m^{*2}}{(e^2\varepsilon^2-A^2)^{1/2}}\right]\cdot\\
&\times\{1+4\pi^2(e^2\varepsilon^2-A^2)^{-1/2}\Delta S_N(R)\}.
\end{aligned}\tag{49}$$

In order to calculate $\Delta S_N(R)$, let us study the roots κ_n of the MIT bag boundary condition (36). If we restrict ourselves to the two lightest flavours u and d, it can be seen from the work of Shankar *et al* (1983) that $m/\kappa_0 \ll 1$ for m of the order of 10 to 100 MeV. Therefore, one can neglect the ratio m/κ in (36). The roots κ_n are then determined by the equation

$$J^2_{\mu+1}(\kappa R) - J^2_\mu(\kappa R) = 0.$$

Using the asymptotic expansions of $J_\mu(\kappa R)$, this equation for large values of R is written as

$$\tan^2(\kappa R - \pi\mu/2 - \pi/4) = 1.$$

The roots of this equation are given by two branches

$$\kappa_n^0 R = (2n + \mu)\pi/2$$

and

$$\kappa_n^0 R = (2n + \mu + 1)\pi/2. \tag{50}$$

That these roots are very accurate can be verified from the exact roots for $\mu = 0$ from Shankar *et al* (1983). These roots are valid for any finite R also as they do not depend on R. Since for integral values of $\mu, J_{-\mu}(\kappa R) = J_\mu(-\kappa R)$, the roots of (n, μ) and $(-n, -\mu)$ are degenerate. It is clear that if $x_n \equiv \kappa_n R$, its approximate value x_n^0 does not depend on R, and therefore $\Delta S_N(R)$ can be written as

$$\Delta S_N(R) = (N/\pi R^2) \sum_{n,\mu} \{\exp[-N\pi x_n^2/R^2(e^2\varepsilon^2 - A^2)^{1/2}] - \exp[-N\pi x_n^{02}/R^2(e^2\varepsilon^2 - A^2)^{1/2}]\}. \tag{51}$$

Since the values of x_n and x_n^0 are very close for low-lying states and that $\{x_n\}$ rapidly converges to $\{x_n^0\}$, the infinite series in (51) is a rapidly convergent series for all values of N and R. The finite volume correction and the MIT bag confinement correction to Schwinger's formula are finally calculated from (49) and (51) using a few low lying states. This is because the contribution of the highly excited states cancels out as for them $x_n \to x_n^0$.

6. Discussion and conclusion

As stated in the introduction, one of the possible ways to produce QGP in an ultrarelativistic heavy ion collision involves breaking down a constant, strong colour electric field ε creating $q\bar{q}$ pairs. In this paper generalization of Schwinger's formula is given when the field exists in finite region and when q and $\bar{q}$ are confined within the plasma region. For a cylindrical geometry, confinement along the radial direction is achieved through the MIT bag boundary condition while along the z-direction a linear confining potential is used. The collision mechanism used is the flux tube model according to which Lorentz contracted receding nuclei are treated as colour charged capacitor plates, creating a configuration of several flux tubes with different colours, whose radius is about 1 fm. The flux tube is of infinite length, confining the field ε in this semi-infinite region. The spontaneous decay of the field into $q\bar{q}$ pairs produces the QGP. In the following, consequences of the results of this paper on QGP formation in ultra-relativistic heavy ion collisions are discussed.

As seen from solutions in (18) and (30), the effect of confinement on QGP formation is that only fields $\varepsilon > A/e$ can produce it. The field of strength $\varepsilon < A/e$ cannot decay spontaneously into $q\bar{q}$ pairs. In the case of an infinite region of constant ε, the critical field strength ε_c for spontaneous breaking down of the field can be determined from (44). From this equation, the rate of pair creation, $P(\varepsilon)$, is

$$P(\varepsilon) \sim \exp[-\pi m^2 e^2 \varepsilon^2/(e^2\varepsilon^2 - A^2)^{3/2}].$$

The critical field ε_c is defined from the relation $P(\varepsilon_c) = \exp(-\pi)$, which gives the following cubic equation

$$(1 - A^2/e^2\varepsilon^2)^3 - m^4/e^2\varepsilon^2 = 0.$$

From this expression it is clear that

$$1 - m^4/e^2\varepsilon^2 = 1 - (1 - A^2/e^2\varepsilon^2)^3 > 0.$$

This inequality implies that $\varepsilon_c(A) > \varepsilon_c(A = 0) = m^2/e$, the critical field without confinement. In the presence of a confining potential higher field strength is required to break the field down into $q\bar{q}$ pairs. It implies that QGP will be formed during the period when the field strength remains $\varepsilon > \max[\varepsilon_c(A), A/e]$. From the above model of the reaction, it will be very early stage of the collision during which this is the case. Thus the confinement of $q(\bar{q})$ favours QGP formation in the early stages of the collision; this tends to decrease the QGP formation time. After this stage of QGP formation, the field strength is reduced to $\varepsilon < \varepsilon_c(A)$, for which pair creation is not possible. During this stage QGP will expand under the influence of the confining potential and weak field ε.

Following Daugherty and Lerche (1976) let us estimate the time required for the field of strength $\varepsilon > \varepsilon_c(A)$ to break down into $q\bar{q}$ pairs. Towards this goal, let us assume that the average energy of the pair is $2mc^2\gamma$ (where γ is a constant). Then the rate of loss of the electric field energy $E(\varepsilon)$ is

$$\frac{\mathrm{d}E(\varepsilon)}{\mathrm{d}t} = 2mc^2\gamma\, P(\varepsilon).$$

From this, the estimate of decay time $\tau(\varepsilon)$ of ε into $q\bar{q}$ pairs is

$$\begin{aligned}\tau(\varepsilon) &= (\varepsilon^2/8\pi)/2mc^2\gamma\, P(\varepsilon)\\ &\simeq (\varepsilon^2/16\pi mc^2\gamma)\exp[\pi m^2 e^2\varepsilon^2/(e^2\varepsilon^2 - A^2)^{3/2}].\end{aligned} \tag{52}$$

For higher field energy $\varepsilon^2/8\pi$, the electric field ε is stronger, and positive argument of the exponential function in (52) is smaller thereby $\tau(\varepsilon)$ is smaller. This implies that the higher the field energy deposited in the collision the shorter is its decay time into $q\bar{q}$ pairs, i.e., shorter is the QGP formation time. From (52), it is seen that nonzero A tends to increase this QGP formation time $\tau(\varepsilon)$. For QGP to be produced in a nuclear collision, one should have $\tau(\varepsilon)$ to be shorter than the nuclear collision time. The number of pairs produced in a volume V per second by this mechanism is

$dN/dt = \tau(\varepsilon)\, V$. Even with all the favourable conditions if not enough $q\bar{q}$ pairs are created from this mechanism, then the alternate mechanism of direct $q\bar{q}$ pair creation from gluons in the presence of a colour electric field will have to be considered (Daugherty and Lerche 1976).

The radial confinement of the plasma by the MIT bag boundary condition on the cylindrical surface introduces the radius (R)-dependent effect on the plasma production mechanism as shown in (49) and (50). On comparison of the values of x_n from Shankar *et al* (1983) with x_n^0, one notices that $x_n \geqslant x_n^0$ and $x_n \to x_n^0$ rapidly as n increases. Therefore, from (51), it follows that $\Delta S_N(R) < 0$, and along with (50), the probability of pair creation $P(\varepsilon, A) < P(\varepsilon, A = 0)$ and is dependent on R. The R-dependent radial confinement effects are present even when $A = 0$. For a given value of ε, they are enhanced for nonzero A. As seen from (50), they decrease when the strength of the colour electric field ε is large. These effects vanish as R approaches infinity since $\Delta S_N(R)$ approaches zero.

Finally, for $A = 0$, the rate of $q\bar{q}$ pair creation in (44) is identical with that derived by Schönfeld *et al* (1990) using the time-dependent vector coupling potential $A^\mu = (0, 0, e\varepsilon t; 0)$.

References

Casher A, Neuberger H and Nussinov S 1979 *Phys. Rev.* **D20** 179

Cox P H and Yidiz A 1985 *Phys. Rev.* **D32** 819

Daugherty J R and Lerche I 1976 *Phys. Rev.* **D14** 340

Dosch H G and Gromes D 1986 *Phys. Rev.* **D33** 1378

Glendenning N K and Matsui T 1983 *Phys. Rev.* **D28** 2890

Gradshteyn I S and Ryzhik I W 1965 *Table of integrals and series and products* (New York: Academic Press) p. 1065

Greiner W, Müller B and Rafelski J 1985 *Quantum electrodynamics of strong fields* (Berlin: Springer-Verlag) ch. 9

Heisenberg W and Euler H 1936 *Z. Phys.* **98** 714

Martin C and Vautherin D 1989 *Phys. Rev.* **D40** 1667

Ru-keng Su and Yuhong Zhang 1984 *J. Phys.* **A17** 851

Schönfeld Th, Schaffer A, Müller B, Sailer K, Reinhardt J and Greiner W 1990 *Phys. Lett.* **B247** 5

Schwinger J 1951 *Phys. Rev.* **82** 664

Shankar R, Vasak D, Warke C S, Greiner W and Müller B 1983 *Z. Phys.* **C18** 327

Wang R C and Wong C Y 1988 *Phys. Rev.* **D38** 348